Impact,
Rheological models,
Unilateral constraints

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MODELLING OF IMPACT PROBLEMS USING RHEOLOGICAL SCHEMES

The main goal of the paper is the analysis of selected impact problems in discrete mechanical systems. We applied rheological structures and unilateral constraints for modelling of interaction between colliding bodies. The rheological structure is a model of elastic-dissipative properties of bodies in impact. It is also a model of energy absorbing devices. The mathematical descriptions of impact problems are presented. Based on the obtained non-linear differential equations we built programs suited for computer simulations. The results of simulations i.e. time histories of displacements and contact forces are presented.

MODELOWANIE ZDERZENIA PRZY UŻYCIU STRUKTUR REOLOGICZNYCH

Głównym celem pracy jest analiza zderzenia w wybranych, dyskretnych układach mechanicznych. Do zamodelowania oddziaływania pomiędzy zderzającymi się ciałami wykorzystano strukturę reologiczną i więzy jednostronne. Struktury reologiczne służą do odwzorowania sprężysto-dyssypacyjnych cech zderzających się ciał lub urządzeń amortyzujących. Sformułowano matematyczne opisy zderzeń. Na podstawie wyprowadzonych nieliniowych równań różniczkowych zbudowano programy do symulacji komputerowej. Wyniki obliczeń przedstawiono w formie przebiegów przemieszczeń i sił kontaktowych.

1. INTRODUCTION

The impact is a short-lived phenomenon of energy exchange between colliding bodies. As a result, the velocities of colliding bodies change rapidly and the reactions are impulsive in nature what means that the interactions are short-lived and reach large values. Within the classical impact theory of rigid bodies it is assumed that the collision phenomenon results with discontinuous change of velocities and the reaction are modeled as impulses. However, such idealization may not be valid in many mechanical problems especially when we need to evaluate the history of reaction within a short time of collision.

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In the model of the impact of two elastic spheres presented by Hertz (see [3] or [5]), the local deformability of these spheres is assumed using a non-linear spring possessing the following property

$$f(x) := k\,x^{3/2} \tag{1}$$

where k denotes a parameter depending on the spheres' radii and material. The Hertz model of impact may be used for modelling of elastic collisions.

In this paper we will apply rheological structures and unilateral constraints for modeling of interaction between colliding bodies. The rheological structure is a model of elastic-dissipative properties of bodies in impact. It is also a model of energy absorbing devices. We will analyze linear viscoelastic structures. Using unilateral constraints one can model contact interactions between bodies. The mathematical descriptions of impact problems will be presented. Based on the obtained differential equations we will show the results of numerical simulations.

2. MATHEMATICAL DESCRIPTION OF IMPACT PROBLEM

Let us consider the description of impact phenomenon in the system visualized in Fig. 1. This system is composed of a body possessing a mass m and of a rheological Kelvin-Voigt structure with parameters k_o and c_o. The unilateral constrains in this system have a form of a bumper visualized in the figure by two horizontal bold lines. We also assume the gravity load acting on the body.

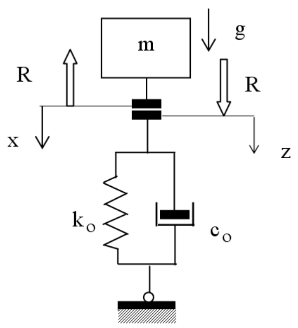

Fig.1. Visualization of the System No. 1.

The coordinates x and z shown in the Fig. 1 determine the displacement of the body and deformation of the rheological structure. In the initial time, for $t = 0$, both bumpers are in contact. The R denotes the force of mutual interaction between the body and the structure. The description of the impact problem for the analyzed system may be obtained by formulation of the equation of the body's motion and evolution of the structure's deformation.

The equations describing the impact problem are as follows

$$m\ddot{x} = mg - R \tag{2a}$$

$$c_o\dot{z} + k_o z = R \tag{2b}$$

$$u := z - x, \quad u \geq 0, \quad R(u - \hat{u}) \geq 0 \; \forall \hat{u} \geq 0 \tag{2c}$$

where Eq. (2a) describes the body motion while Eq. (2b) represents the evolution of the structure's deformation. The system of Eqs. (2c) defines the relationships between a coordinate $u := z - x$ and the reaction R. The relations (2c) are visualized in Fig. 2.

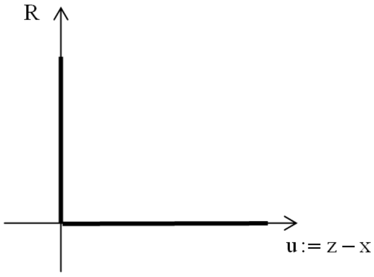

Fig.2. Graph of the mapping expressed via Eqs. (2c).

Based on the relations (2c) it may be proved that for such time instants when $u = 0$, we can formulate so called differential successions (see [1], [2] and [4]) as a relationship between R and $\dot{u}$ to be shown in Fig. 3.

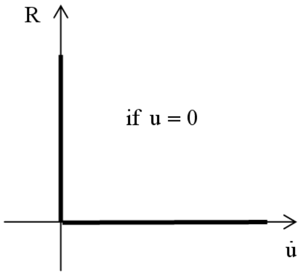

Fig.3. Differential successions of Eqs. (2c) for u=0.

Applying the mathematical formulation of the impact problem presented above it is possible to obtain the explicit system of non-linear equations

$$m\ddot{x} = mg - R \tag{3a}$$

$$\dot{z} = \begin{cases} -\dfrac{k_o}{c_o} z & \text{if} \quad z - x > 0 \quad \text{or} \quad z - x = 0 \;\text{ and }\; c_o \dot{x} + k_o x \leq 0 \\ \dot{x} & \text{if} \quad z - x = 0 \;\text{ and }\; c_o \dot{x} + k_o x > 0 \end{cases} \tag{3b}$$

$$R = \begin{cases} 0 & \text{if} \quad z - x > 0 \\ \left[c_o \dot{x} + k_o x\right]^{+} & \text{if} \quad z - x = 0 \end{cases} \tag{3c}$$

completed by the following initial conditions

$$x(0)=0, \quad \dot{x}(0)=v_o, \quad z(0)=0 \tag{3d}$$

The function $[\cdot]^+$ used in Eq. (3c) denotes projection onto the set of non-negative numbers

$$[\xi]^+ := \begin{cases} 0 & \text{if} \quad \xi \le 0 \\ \xi & \text{if} \quad \xi > 0 \end{cases} \tag{4}$$

Let us analyze another example shown in Fig. 4. Similarly to Eqs. (2) we can formulate the following equations

$$m\ddot{x} = mg - R \tag{5a}$$

$$k(y-x) = -R \tag{5b}$$

$$c_o\dot{z} + k_o z = R \tag{5c}$$

$$u := z - y, \quad u \ge 0, \quad R(u-\hat{u}) \ge 0 \; \forall \hat{u} \ge 0 \tag{5d}$$

$$x(0)=0, \quad \dot{x}(0)=v_o>0, \quad y(0)=0, \quad z(0)=0 \tag{5e}$$

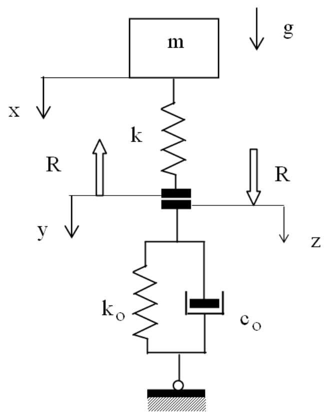

Fig.4. Visualization of the System No. 2.

A graph illustrating the relationships (5d) is the same as it was shown in Fig. 2. The system of Eqs. (5) can be re-written to such a form

$$m\ddot{x} = mg - R \tag{6a}$$

$$\dot{y} = \begin{cases} \dot{x} & \text{if} \quad z > y \\ \dot{x} & \text{if} \quad z = y = x \text{ and } \dfrac{k_o}{c_o}z + \dot{x} \le 0 \\ \dot{z} & \text{if} \quad z = y = x \text{ and } \dfrac{k_o}{c_o}z + \dot{x} > 0 \\ \dot{z} & \text{if} \quad z = y < x \end{cases} \tag{6b}$$

$$\dot{z} = \begin{cases} -\dfrac{k_o}{c_o} z & \text{if} \quad z > y \\ \dfrac{1}{c_o}\left[-\left(k_o + k\right) z + k x\right] & \text{if} \quad z = y \end{cases} \tag{6c}$$

$$R = \begin{cases} 0 & \text{if} \quad z > y \\ k\left(x - z\right) & \text{if} \quad z = y \end{cases} \tag{6d}$$

The last impact problem we will analyze is presented in Fig. 5. Based on this figure we formulate the following equations

$$m\ddot{x} = mg - R \tag{7a}$$

$$c\left(\dot{y} - \dot{x}\right) + k\left(y - x\right) = -R \tag{7b}$$

$$c_o \dot{z} + k_o z = R \tag{7c}$$

$$u := z - y, \quad u \geq 0, \quad R\left(u - \hat{u}\right) \geq 0 \ \forall \hat{u} \geq 0 \tag{7d}$$

$$x(0) = 0, \quad \dot{x}(0) = v_o > 0, \quad y(0) = 0, \quad z(0) = 0 \tag{7e}$$

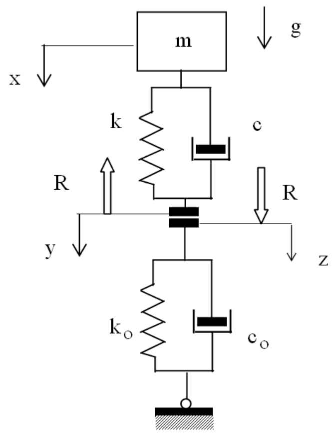

Fig.5. Visualization of the System No. 3.

The illustration of relationships (5d) is given in Fig. 2 while the differential successions are presented in Fig. 3. After some algebra we obtain the following equations

$$m\ddot{x} = mg - R \tag{8a}$$

$$\dot{y} = \begin{cases} \dot{x} + \dfrac{k}{c}\left(x - y\right) & \text{if} \quad z - y > 0 \\ \dot{x} + \dfrac{k}{c}\left(x - y\right) - \dfrac{c_o}{c + c_o}\left[A\right]^{+} & \text{if} \quad z - y = 0 \end{cases} \tag{8b}$$

$$\dot{z} = \begin{cases} -\frac{k_o}{c_o} z & \text{if} \quad z - y > 0 \\ -\frac{k_o}{c_o} z + \frac{c}{c + c_o} [A]^+ & \text{if} \quad z - y = 0 \end{cases} \tag{8c}$$

$$R = \begin{cases} 0 & \text{if} \quad z - y > 0 \\ \frac{c c_o}{c + c_o} [A]^+ & \text{if} \quad z - y = 0 \end{cases} \tag{8d}$$

$$A := \dot{x} + \frac{k}{c}(x - y) + \frac{k_o}{c_o} z \tag{8e}$$

The relationships expressed via Eqs. (3) and (6) and (8) will be used for computer simulation of impact problems in analyzed systems.

3. SIMULATIONS OF IMPACT PROBLEMS

Computer simulations were carried out for three analyzed systems. The parameters of these systems are presented in Tab. 1.

Tab. 1. Parameters of analyzed systems

m [kg]	k [N/m]	k_o [N/m]	c [N·s/m]	c_o [N·s/m]
100	1962	3924	125	250

We assume that a body of mass m falls from rest through a vertical distance of $h_o = 5\,\text{m}$ in the earth's gravitational field. The velocity of the impacting body is equal $v_o = \sqrt{2 g h_o} \approx 10\,\text{m/s}$. The solutions of non-linear differential equations defining the impact problems were obtained applying 4$^{\text{th}}$ order Runge-Kutta method.

Figures 6, 7 and 8 show time history of reaction force in three analyzed systems. The biggest value of the force was obtained for System No. 1 (see Fig. 6) while the smallest one for System No. 3 (see Fig. 8).

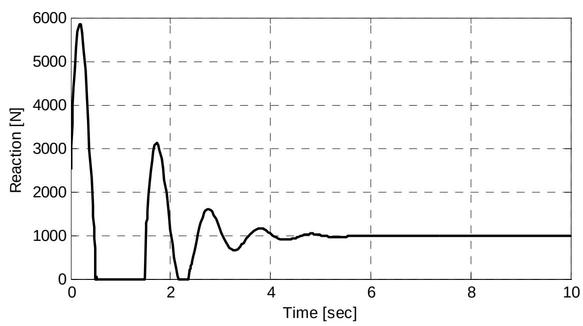

Fig.6. Time history of reaction force in System No. 1 (see Fig. 1).

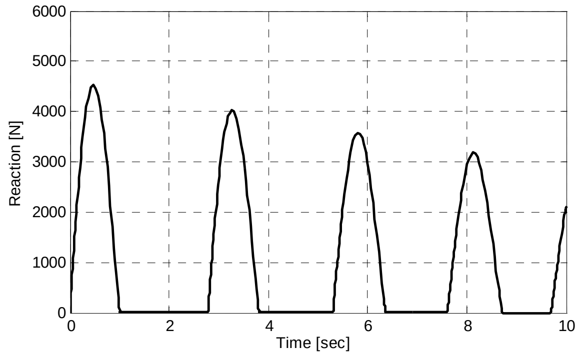

Fig.7. Time history of reaction force in System No. 2 (see Fig. 4).

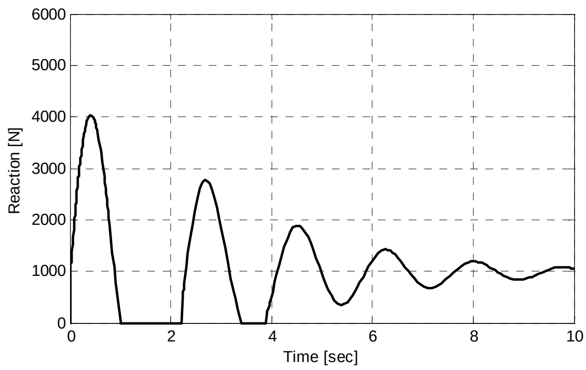

Fig.8. Time history of reaction force in System No. 3 (see Fig. 5).

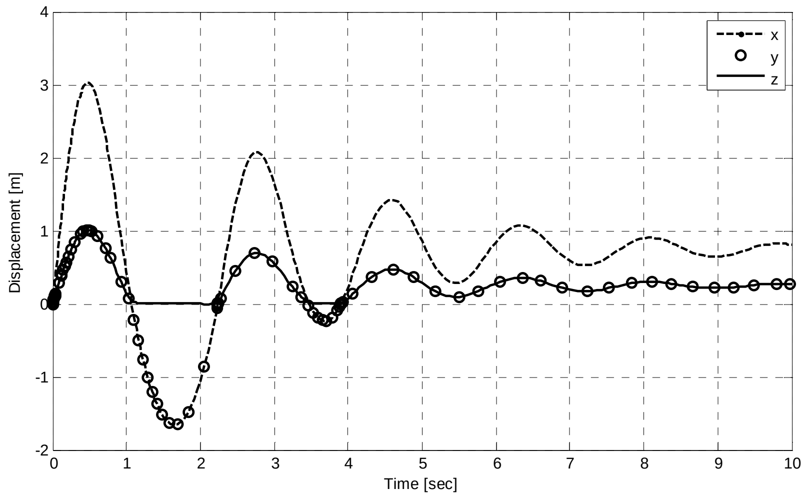

Fig.9. Time history of displacements in System No. 3 (see Fig. 5).

As an example of displacement results we presented only the time history curves for the System No. 3. These results are visualized in Fig. 9 where the evolutions of coordinates x, y and z are given (compare Fig. 5).

4. CONCLUSIONS

The mathematical descriptions and the computer simulations of three impact problems were presented in the paper. The main goal of the work was to formulate the problem of motion for systems possessing unilateral constraints. Our next papers will be devoted to analysis of impact problems for systems containing rheological structures described by integral operators. Non-linear rheological schemes with dry friction elements will be also analyzed. The results of such problems will be applied for modelling of rigid body impact into elastic-plastic deformable continuum structure.

5. REFERENCES

[1] Grzesikiewicz W.: *Dynamika układów mechanicznych z więzami,* Warszawa, WPW 1990.
[2] Grzesikiewicz W., Wakulicz A., Zbiciak A.: *Succession of constraint imposed on time function*. Polioptymalizacja i Komputerowe Wspomaganie Projektowania, tom 7, s. 49-56, Wyd. Uczelniane Politechniki Koszalińskiej, Koszalin 2009.
[3] Stronge J. W.: *Impact Mechanics,* Cambridge Univ. Press, 2000.
[4] Zbiciak A.: *Dynamika materiałów i konstrukcji o nieklasycznych charakterystykach sprężysto-dyssypacyjnych.* Warszawa, OWPW, 2010.
[5] Пановко Я. Г.: *Введение в теорию механического удара*, Наука, Moscow 1977.