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Aleksander SZWED[1]

DISSIPATION FUNCTIONS AND YIELD CONDITIONS FOR GEOLOGICAL MATERIALS WITH INTERNAL CONSTRAINTS

General framework of modeling plasticity in geological materials with internal constraints is discussed in this paper. A family of plastic dissipation functions devoted to soil and rock mechanics is proposed. Present paper is a generalization of Drucker-Prager model and earlier author's papers, which has dealt with incompressible metals. The proposed functions are dependent on the three invariants of the plastic strain rate tensor and material parameters. In the space of principal plastic strain rates the curves of constant dissipation have three axes of symmetry in the deviatoric plane. Internal kinematical constraints in the material are utilized. Using the potential constitutive law the constitutive relation for material is derived. Obtained yield surfaces have a conical shape in the spectral stress space. The failure surfaces have three axes of symmetry in the deviatoric plane cross-sections. The deviatoric cross-section curves of the failure surface may change from equilateral triangle through the circle and then to the equilateral triangle oriented in the opposite way.

FUNKCJE DYSSYPACJI I WARUNKI PLASTYCZNOŚCI MATERIAŁÓW GEOLOGICZNYCH Z WIĘZAMI WEWNĘTRZNYMI

Zaproponowano funkcję dyssypacji przeznaczoną do modelowania plastyczności w gruntach i skałach będącą uogólnieniem modelu Druckera-Pragera. Funkcja zależy od trzech niezmienników tensora prędkości odkształcenia plastycznego i parametrów materiałowych. Funkcję dyssypacji stosuje się do wyznaczenia relacji konstytutywnej materiału idealnie plastycznego z więzami wewnętrznymi oraz warunku plastyczności. Przekroje dewiatorowe powierzchni plastyczności mogą zmieniać się od trójkątnego do kołowego, zaś południki są prostoliniowe.

1. INTRODUCTION

The deformation and strength characteristic of geological materials such as sands, clay and rock are usually modeled within the framework of small strains and rate-independent elastoplasticity [1,4]. This type of constitutive modeling is based on the additive split of

[1]Warsaw University of Technology, Civil Engineering Faculty, POLAND, Warszawa 00–637, Armii Ludowej 16.
Phone: 48 22 234-56-76, e–mail: a.szwed@il.pw.edu.pl

strain rate tensor into elastic and plastic parts, $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_e + \dot{\boldsymbol{\varepsilon}}_p$ [2,7]. The elastic part $\dot{\boldsymbol{\varepsilon}}_e$ is governed by Hooke's law. The plastic part of strains $\dot{\boldsymbol{\varepsilon}}_p$ can be defined using concept of the yield function (or plastic potential) given in stresses, or by the plastic dissipation function, which is a function of plastic strain rate tensor [4,5]. Perfectly plastic model based on the dissipation function for isotropic material is analyzed in details in this paper. In the literature related to modeling plasticity, there is a very few simple models based on dissipation function approach [4,6].

2. INTERNAL CONSTRAINTS IN MATERIAL

In case of modeling of plasticity in geological materials there is a need to introduce internal constraints. Constraints can be used to model incompressibility [8], or develop dual formulation for linear meridians [5,6], or even more in case of lack of the associative plastic flow in the frictional materials [3]. Derivation of constitutive models based on the Drucker-Prager and the Coulomb-Mohr yield conditions require usage of the second type of constraints from the mentioned. Formulation of basic equations for this type of models is discussed in the following.

Dissipation function suitable for isotropic material is regarded in a general form,

$$D\left(p, q, \cos 3\phi\right) = 0 \tag{1}$$

or in case of incompressible material can be modified to $D_{IN}\left(q, \cos 3\phi\right)$, [8]. General form of the constraint function can be of the form:

$$W\left(p, q, \cos 3\phi\right) = 0\,. \tag{2}$$

The following invariants of plastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}_p$ are used throughout the paper:

$$p = \frac{1}{\sqrt{3}}\,\mathrm{tr}\dot{\boldsymbol{\varepsilon}}_p, \qquad q = \sqrt{\mathrm{tr}\dot{\mathbf{e}}_p^2}, \qquad \cos 3\phi = \frac{\sqrt{6}\,\mathrm{tr}\dot{\mathbf{e}}_p^3}{\sqrt{\mathrm{tr}^3\dot{\mathbf{e}}_p^2}}, \quad \text{where} \quad \dot{\mathbf{e}}_p = \dot{\boldsymbol{\varepsilon}}_p - \frac{p}{\sqrt{3}}\mathbf{I} \tag{3}$$

is the deviator of $\dot{\boldsymbol{\varepsilon}}_p$. Constitutive relationship of perfectly plastic material with internal constraints is defined using Lagrange potential in the form,

$$P\left(p, q, \cos 3\varphi\right) = D_{IN}\left(q, \cos 3\varphi\right) - \mu W\left(p, q, \cos 3\varphi\right), \tag{4}$$

in which μ is the Lagrange multiplier. Stress tensor $\boldsymbol{\sigma}$ is then given by the formula,

$$\boldsymbol{\sigma} = \frac{\partial P}{\partial \dot{\boldsymbol{\varepsilon}}_p} = \frac{\partial D_{IN}}{\partial \dot{\boldsymbol{\varepsilon}}_p} - \mu \frac{\partial W}{\partial \dot{\boldsymbol{\varepsilon}}_p} \qquad \text{if} \qquad \dot{\boldsymbol{\varepsilon}}_p \neq \mathbf{0} \tag{5}$$

which, by usage of (4), leads to the following constitutive relationship:

$$\boldsymbol{\sigma} = \alpha_2\,\mathbf{g}_2 + \alpha_3\,\mathbf{g}_3 - \mu\left(\gamma_1\,\mathbf{g}_1 + \gamma_2\,\mathbf{g}_2 + \gamma_3\,\mathbf{g}_3\right). \tag{6}$$

Derivatives of the invariants $(p, q, \cos 3\phi)$, given by (3), with respect to the plastic strain tensor define tensor generators,

$$\mathbf{g}_1 = \frac{\partial p}{\partial \dot{\boldsymbol{\varepsilon}}_p} = \frac{1}{\sqrt{3}}\mathbf{I}, \quad \mathbf{g}_2 = \frac{\partial q}{\partial \dot{\boldsymbol{\varepsilon}}_p} = \frac{1}{q}\dot{\mathbf{e}}_p, \quad \mathbf{g}_3 = \frac{\partial \cos 3\phi}{\partial \dot{\boldsymbol{\varepsilon}}_p} = \frac{3\sqrt{6}}{q^3}\left(\dot{\mathbf{e}}_p^2 - \frac{q\cos 3\phi}{\sqrt{6}}\dot{\mathbf{e}}_p - \frac{q^2}{3}\mathbf{I}\right). \quad (7)$$

Derivatives of the dissipation function and constraint function according to (5) are:

$$\alpha_2 = \frac{\partial D_{IN}}{\partial q}, \quad \alpha_3 = \frac{\partial D_{IN}}{\partial \cos 3\phi}, \quad \text{and} \quad \gamma_1 = \frac{\partial W}{\partial p}, \quad \gamma_2 = \frac{\partial W}{\partial q}, \quad \gamma_3 = \frac{\partial W}{\partial \cos 3\varphi}. \quad (8)$$

Calculating trace in (6) we get the reaction stress to introduced constraints,

$$\gamma_1 \mu = -\frac{1}{\sqrt{3}}\mathrm{tr}\boldsymbol{\sigma} = -\xi. \quad (9)$$

When $\gamma_1 = 1$, we have simplification to $\mu = -\xi$, which leads to convenient form of kinematic constraints to be introduced in the next sections. Deviatoric part of relation (6) is,

$$\mathbf{s} = (\alpha_2 - \mu\gamma_2)\mathbf{g}_2 + (\alpha_3 - \mu\gamma_3)\mathbf{g}_3. \quad (10)$$

The stress tensor function (6) or (10) is a homogeneous function degree zero with regard to the plastic strain rate tensor, and thus this condition implies a failure criterion of the form: $f(\boldsymbol{\sigma}) = 0$, [5,7].

3. DRUCKER-PRAGER MODEL

One of the simplest models used in modeling of plasticity in geological materials is the classical Drucker-Prager (DP) model [4,5]. In this point we formulate the DP model using concept of dissipation function, and then in the next section we propose its generalization to cover wider range of possible material behaviors [1,6].

Nonnegative, convex and homogeneous degree one with respect to the plastic strain rate tensor dissipation function is as follows,

$$D(q) = Bq, \quad (11)$$

where $B > 0$ is a material parameter. In literature one can find different mathematical forms of dissipation function, but all of them are equivalent [5]. Internal kinematic constraints in case of DP material are defined as,

$$W(p,q) = p - Aq = 0, \quad (12)$$

with $A \geq 0$, being material constant.

In case of this material with internal constraints, the potential for determination of stresses is given by the simplified formula,

$$P(p,q) = D(q) - \mu W(p,q) = Bq - \mu(p - Aq), \quad (13)$$

where μ is the Lagrange multiplier. Differentiation of function (13) results in,

$$\boldsymbol{\sigma} = \frac{\partial P}{\partial \dot{\boldsymbol{\varepsilon}}_p} = B\left(\frac{1}{q}\dot{\mathbf{e}}_p\right) - \mu\left(\frac{1}{\sqrt{3}}\mathbf{I} - A\frac{1}{q}\dot{\mathbf{e}}_p\right). \tag{14}$$

The following invariants of the stress tensor $\boldsymbol{\sigma}$ and the stress deviator $\mathbf{s}$ are used throughout the paper:

$$\xi = \frac{1}{\sqrt{3}}\mathrm{tr}\boldsymbol{\sigma}, \qquad r = \sqrt{\mathrm{tr}\mathbf{s}^2}, \qquad \cos 3\Theta = \frac{\sqrt{6}\mathrm{tr}\mathbf{s}^3}{\sqrt{\mathrm{tr}^3\mathbf{s}^2}}, \qquad \text{where} \qquad \mathbf{s} = \boldsymbol{\sigma} - \frac{1}{\sqrt{3}}\xi\mathbf{I} = \boldsymbol{\sigma} - \mathbf{k}. \tag{15}$$

Splitting (14) into the cubic and deviatoric parts we get,

$$\mu = -\xi, \qquad \mathbf{s} = (B - A\xi)\frac{1}{q}\dot{\mathbf{e}}_p, \tag{16}$$

and squaring (16), and then calculating trace, one can obtain the DP yield condition,

$$f(\boldsymbol{\sigma}) = r + A\xi - B = 0, \tag{17}$$

compare [5]. Surface defined by equation (17) is a conical surface of revolution, with axis being the hydrostatic pressure axis ξ.

Graphical interpretation of the yield condition (17) and the dissipation function (11) with the constraints (12) is shown in fig.1. Dual spaces of the plastic strain rate $\dot{\boldsymbol{\varepsilon}}_p$ and the stress tensor $\boldsymbol{\sigma}$ by means of invariants (p,q) and (ξ,r) are given in the fig.1. Meaning of the material parameters A and B is also explained. In the space (p,q) constant dissipation is represented by one point, and in spectral space is represented by a circle in the deviatoric plane for $p = AD/B$, fig.1. Graphical interpretation of the potential law (14) in plastic strain rate space and tangential stress space are also shown. Original and dual spaces highlight geometrical sense of the concept of constitutive relationship formulation, dissipation, constraints and yield condition.

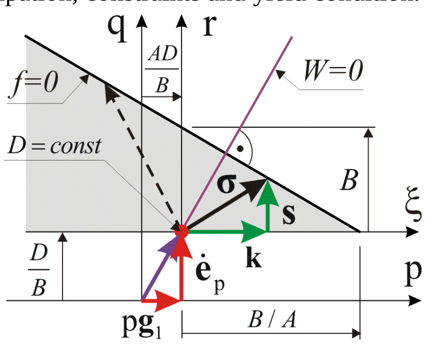

Fig.1. Graphical interpretation of dissipation function, its gradient and dual (plastic strain rate and stress) spaces, as well as, kinematic constraints for Drucker-Prager constitutive relationship

When $A = 0$ condition (17) coincides with Huber-Mises yield condition: $f(\boldsymbol{\sigma}) = r - B = 0$, while constraints define incompressibility: $W(p) = p = 0$, but dissipation function remains the same as (11), [8].

4. GENERALIZATION OF DRUCKER-PRAGER MODEL

Generalization of dissipation function (11) can be proposed in the following form,

$$D(q, \cos 3\phi) = Bq\cos\psi\,, \qquad \text{where} \qquad \psi = \frac{1}{3}\arccos(\alpha\cos 3\phi)\,. \tag{18}$$

Because of non-negativity requirement the size parameter must be $B > 0$. The shape parameter α must be in the range $|\alpha| \le 1$ to preserve convexity of the dissipation function. Graphs of the dissipation (18), represented as $D(q,\phi)$ for the selected values of α are shown in fig.2. The dissipation function defines a cone type surfaces with the vertex at $D(0,\phi) = 0$. The shape of constant dissipation curves in the deviatoric plane of spectral space of $\dot{\boldsymbol{\varepsilon}}_p$ is governed by the shape parameter. For practical use in mechanics of geological materials the shape parameter should be negative in (18).

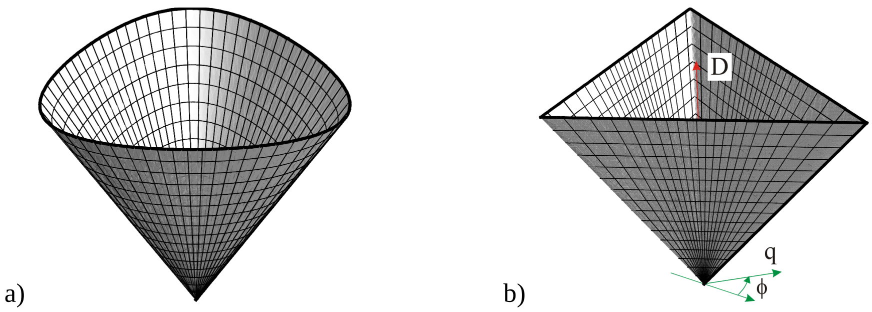

Fig.2. Cones of the dissipation function for: a) $\alpha = -0.5$ *, b)* $\alpha = -1$

Internal constraints in frictional material are assumed in the following form:

$$W(p, q, \cos 3\phi) = p - A\,q\cos\psi = 0\,, \tag{19}$$

where $A \ge 0$ being the slope parameter, whilst ψ is defined in (18). Graphs of the constraints (19) in the space of kinematical invariants (p, q, ϕ) are cones with vertex located in the point $(p;q) = (0;0)$, and A defines the slope of cones. Cone axis is the p-axis, while the shape of deviatoric cross-section is governed by α.

Potential for the proposed material description with internal constraints is defined as,

$$P(p, q, \cos 3\phi) = B\,q\cos\psi - \mu(p - A\,q\cos\psi)\,. \tag{20}$$

The constitutive relationship of perfectly plastic material is:

$$\boldsymbol{\sigma} = \frac{\partial P}{\partial \dot{\boldsymbol{\varepsilon}}_p} = \alpha_2\,\mathbf{g}_2 + \alpha_3\,\mathbf{g}_3 - \mu(\gamma_1\,\mathbf{g}_1 + \gamma_2\,\mathbf{g}_2 + \gamma_3\,\mathbf{g}_3)\,. \tag{21}$$

Derivatives of the dissipation function (18) and constraints function (19) with respect to the invariants $\left(p,q,\cos 3\phi\right)$ are:

$$\alpha_2 = B\cos\psi\,,\quad \alpha_3 = Bq\frac{\alpha\sin\psi}{3\sin 3\psi}\,,\quad \text{and}\quad \gamma_1 = 1\,,\quad \gamma_2 = -A\cos\psi\,,\quad \gamma_3 = -Aq\frac{\alpha\sin\psi}{3\sin 3\psi}\,. \tag{22}$$

Calculating trace of equation (21) one can get the reaction stress, $\xi = -\mu$, for the applied internal constraints. Multiplier μ can be evaluated from a boundary valued problem [5]. Using obtained results the constitutive relationship for deviatoric part of (21) can be expressed as:

$$\mathbf{s} = C\cos\psi\,\mathbf{g}_2 + C\,q\frac{\alpha\sin\psi}{3\sin 3\psi}\mathbf{g}_3\,, \tag{23}$$

where $C(\xi) = B - A\xi$ is a linear function of the stress invariant ξ.

Using the deviatoric part (23) of constitutive relationship for derivation of the stress invariants r and $\cos 3\Theta$, defined in (15), we get the following results:

$$\frac{r^2}{C^2} = 1 - \frac{1-\alpha^2}{\left(1+2\cos 2\psi\right)^2}\,,\qquad 2\alpha\frac{r^3}{C^3}\cos 3\Theta = 1+\alpha^2 - \frac{3\left(1-\alpha^2\right)}{\left(1+2\cos 2\psi\right)^2} - \frac{2\left(1-\alpha^2\right)^2}{\left(1+2\cos 2\psi\right)^3}\,. \tag{24}$$

In equations (24), invariant ψ is the kinematical parameter, which can be eliminated. Elimination of ψ allows for expression of the yield condition as a function of the stress invariants and material constants,

$$f(\boldsymbol{\sigma}) = \left[3C\,r^2 - 3\left(2-\alpha^2\right)C^3 - 2\alpha\,r^3\cos 3\Theta\right]^2 - 4\left(1-\alpha^2\right)\left(C^2 - r^2\right)^3 = 0\,. \tag{25}$$

Explicit form of the yield condition expressed by stress invariants and simultaneously explicit form of the dissipation function can be obtained only in case of very simple models, see [1,4,5,6] for further details. The presented model involves the first, second and third invariants where the formulation of equations is in explicit mathematical form. The constant dissipation curve and yield surface, $f(\boldsymbol{\sigma}) = 0$, are shown in fig.3 and fig.4.

In case of $\alpha = -1$, yield condition (25) coincides with the generalized Rankine (R) yield criterion, $f(\boldsymbol{\sigma}) = 2r^3\cos 3\Theta + 3\left(B - A\xi\right)r^2 - \left(B - A\xi\right)^3 = 0$. When $\alpha = 0$, the Drucker-Prager yield condition can be obtained from (25), $f(\boldsymbol{\sigma}) = 2r - \sqrt{3}\left(B - A\xi\right) = 0$. In the range $-1 \le \alpha \le 0$, curves shown in fig.4 can change continuously from the R to DP yield conditions.

It is convenient to use another shape parameter with clear graphical interpretation in the stress space instead of the shape parameter α, see fig.3b. The shape parameter of yield condition in the deviatiric cross-section is defined as:

$$t=\frac{r_C}{r_T}=\frac{q_T}{q_C}, \qquad \text{then} \qquad \alpha(t)=\frac{3\sqrt{3}(1-t)t}{2\sqrt{\left(t^2-t+1\right)^3}}. \tag{26}$$

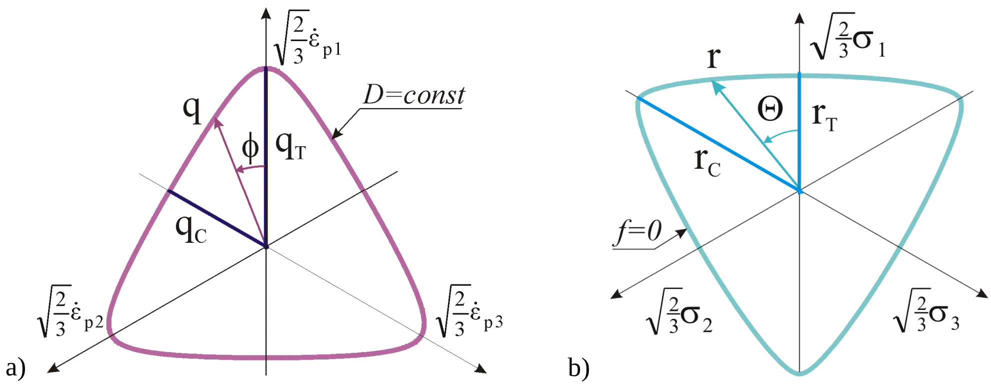

Fig.3. (a) Dissipation function; (b) deviatoric cross-section of yield condition for $t=1.6$

Parameters $q_T=q(0)$ and $q_C=q(\pi/3)$ used in (26) are values of the second invariant of plastic strain rate deviator calculated from dissipation (18), $q(\phi)=D_C/(B\cos\psi)$, where D_C is constant value of dissipation. Interpretation of the stress and strain rate invariants used in both spaces is given in fig.3. Parameters r_T and r_C used in (26) are values of the second stress deviator invariant obtained from criterion (25). Sub-index T stands for the tension meridian: $\Theta=0$ $(\sigma_1\geq\sigma_2=\sigma_3)$ and sub-index C means the compression meridian: $\Theta=\pi/3$ $(\sigma_1=\sigma_2\geq\sigma_3)$ of the yield condition. From fig.3 it can be seen, that $\phi=0$ and $\phi=\pi/3$ are axes of symmetry of the dissipation function graphs, while $\Theta=0$ and $\Theta=\pi/3$ are axes of symmetry of the yield condition graphs in the deviatoric plane. Material parameters A, B and $1/2\leq t\leq 2$ can be derived using typical experimental tests: uniaxial tension and compression, biaxial compression. The shape parameter t is then expressed by the yield stresses.

5. CONCLUSIONS

The perfectly plastic material model based on the dissipation function was proposed. The dissipation function is useful in modeling plastic properties of isotropic geological materials [2]. Failure criterion have desired properties: three axes of symmetry in the deviatoric plane and shape may change between equilateral triangle and circle. Determination of parameters from experimental tests is simple to perform. A wide range of possible change of the material parameters preserving convexity can cover majority of the material behaviors [1,4,6]. Graphs of the yield conditions in the deviatoric cross-section for several values of the shape parameter $t\in[1,2]$ are shown in fig.4.

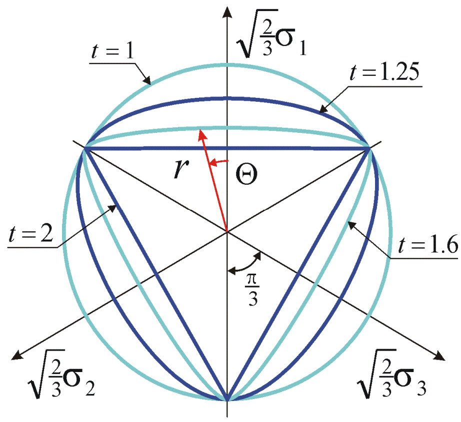

Fig.4. Possible changes in shape of the yield condition in deviatoric cross-section

The generalization of DP model is proposed. The dissipation and the yield criterion are expressed in analytical explicit form - a very few in the literature is available. Further generalization of the proposed model can be done. The proposed dissipation function and the yield condition can be applied to the limit analysis and computer applications for soil and rock mechanics.

6. REFERENCES

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