masonry structures, elasticity, plasticity, non-homogeneous materials, orthotropy, invariants , finite element method, user subroutines, ABAQUS

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MODELLING OF ORTHOTROPIC MASONRY STRUCTURES

A class of constitutive models of orthotropic non-homogeneous masonry materials in the framework of elasto-plasticity theory of small displacements with energetic hardening/softening is proposed. Analysed class of constitutive models reduces to the classical one for ideal elasto-plasticity with well-known Hoffman yield condition for orthotropic homogeneous materials. Proposed class of models is implemented in FEM system ABAQUS. Constitutive relationships are implemented in FORTRAN in user procedure UMAT. The numerical tests are proposed to check correctness of the implementation, and some boundary value problems are also solved.

MODELOWANIE ORTOTROPOWYCH KONSTRUKCJI MUROWYCH

W artykule przedstawiono sformułowanie teorii sprężysto plastyczności niejednorodnych ortotropowych materiałów murowych z tzw. energetycznym wzmocnieniem/osłabieniem. Analizowana klasa modeli redukuje się do znanych modeli idealnej sprężysto-plastyczności w warunkiem plastyczności Hoffmana. Zaproponowaną klasę modeli zaimplementowano w programie MES ABAQUS. Relacje konstytutywne zaprogramowano w języku FORTRAN w procedurze użytkownika UMAT. Przydatność zaimplementowanego modelu zaprezentowano na przykładzie zadania brzegowego ścinania ściany z otworem.

1. INTRODUCTION

This work confirms usefulness of the concept of structural tensors in masonry mechanics for a formulation of orthotropic failure criteria and constitutive relationships of elasto-plasticity theory. The theory of representation of orthotropic scalar-valued tensor functions and the convex analysis have been used, cf. [2]. From theoretical point of view,

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the orthotropic failure criterion is a surface, bounding a convex set in the space of the stress tensor. It is a scalar-valued function dependent on seven invariants of structural tensors and the stress tensor and has to satisfy the convexity requirement for all stress conditions.

In order to capture properly the entire range of the stress states corresponding to the two distinct zones (being tension and compression parts), the proposed 3D orthotropic plastic criterion for a masonry material is represented by two quadratic functions of the invariants. On the whole, it includes 15 independent material constants. These material parameters are dependent of peak stress values obtained from appropriate uniaxial and biaxial tests. This criterion can be treated as a generalization of the well known Hoffman failure criterion which was originally formulated as a quadratic function of stress components utilizing nine independent material parameters, cf. [4]. In the simplest situation, when only one failure criterion is used, 12 parameters is required. Then, the Hoffman criterion of 9 independent parameters may be regarded as its special case. The proposed model was implemented in frame of user subroutine UMAT in ABAQUS software. For such implementation tests for homogenous stress and strain fields were carried out showing correctness of the subroutine.

2. INVARIANT FORMULATION OF A NEW CLASS OF YIELD AND FAILURE CONDITIONS

In the invariant formulation the two yield functions can be written in the following form:

$$f_1\left(K_p\right)-\left(1+L_1\kappa_1\right)=0 \text{ and } f_2\left(K_p\right)-\left(1+L_2\kappa_2\right)=0, \tag{1}$$

where L_α are given constant plastic parameters, κ_α are internal hardening variables, and

$$f_\alpha\left(K_p\right)=a_i^{(\alpha)}K_i+b_{ij}^{(\alpha)}K_iK_j+c_i^{(\alpha)}K_{i+3}. \tag{2}$$

Material parameters: $a_i^{(\alpha)}$, $b_{ij}^{(\alpha)}$ and $c_i^{(1)}=c_i^{(2)}>0$ must be determined from uniaxial and biaxial strength experimental data. The invariants K_p $\left(p=1,\ldots,6\right)$ are given by:

$$\begin{aligned}
&K_1=tr\mathbf{M}_1\boldsymbol{\sigma}, \qquad K_2=tr\mathbf{M}_2\boldsymbol{\sigma}, \qquad K_3=tr\mathbf{M}_3\boldsymbol{\sigma},\\
&K_4=\left(tr\mathbf{M}_1\boldsymbol{\sigma}\right)^2-\left(tr\mathbf{M}_2\boldsymbol{\sigma}\right)^2-\left(tr\mathbf{M}_3\boldsymbol{\sigma}\right)^2-tr\mathbf{M}_1\boldsymbol{\sigma}^2+tr\mathbf{M}_2\boldsymbol{\sigma}^2+tr\mathbf{M}_3\boldsymbol{\sigma}^2,\\
&K_5=\left(tr\mathbf{M}_2\boldsymbol{\sigma}\right)^2-\left(tr\mathbf{M}_1\boldsymbol{\sigma}\right)^2-\left(tr\mathbf{M}_3\boldsymbol{\sigma}\right)^2-tr\mathbf{M}_2\boldsymbol{\sigma}^2+tr\mathbf{M}_1\boldsymbol{\sigma}^2+tr\mathbf{M}_3\boldsymbol{\sigma}^2,\\
&K_6=\left(tr\mathbf{M}_3\boldsymbol{\sigma}\right)^2-\left(tr\mathbf{M}_1\boldsymbol{\sigma}\right)^2-\left(tr\mathbf{M}_2\boldsymbol{\sigma}\right)^2-tr\mathbf{M}_3\boldsymbol{\sigma}^2+tr\mathbf{M}_1\boldsymbol{\sigma}^2+tr\mathbf{M}_2\boldsymbol{\sigma}^2,
\end{aligned} \tag{3}$$

where $\mathbf{M}_j=\mathbf{m}_j\otimes\mathbf{m}_j$ (no summation over *j*, and $\mathbf{m}_i\cdot\mathbf{m}_j=\delta_{ij}$) and vectors $\mathbf{m}_i$ are vectors of principal axes of orthotropy. Function f_α is convex with respect to $\boldsymbol{\sigma}$ if and only if the 3×3 matrix $\left[b_{ij}^{(\alpha)}\right]$ is semi-positive definite. We construct the failure limit in such a way that the following set:

$$B = B_1 \cap B_2 \quad \wedge \quad B_\alpha \equiv \left\{ \boldsymbol{\sigma} \in T_2^s \,\middle|\, f_\alpha\left(K_p\right) - 1 < 0 \right\}, \tag{4}$$

is convex. We also assume that at least the function f_1 is convex with respect to $\boldsymbol{\sigma}$.

3. CONSTITUTIVE RELATIONSHIPS

The elastic-plastic orthotropic material is considered with the assumption of an additive decomposition of the strain tensor

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \;, \tag{5}$$

into the elastic part $\boldsymbol{\varepsilon}^e$ and the plastic part $\boldsymbol{\varepsilon}^p$. The elastic part is defined by the orthotropic Hooke's law:

$$\boldsymbol{\sigma} = \left.\frac{\partial W_S}{\partial \boldsymbol{\varepsilon}^e}\right|_{\boldsymbol{\varepsilon}^e = \left(\boldsymbol{\varepsilon}^e\right)^T} = \left.\frac{\partial W_S}{\partial \boldsymbol{\varepsilon}^e \otimes \partial \boldsymbol{\varepsilon}^e}\right|_{\boldsymbol{\varepsilon}^e = \left(\boldsymbol{\varepsilon}^e\right)^T} .\boldsymbol{\varepsilon}^e \equiv \mathbf{C}.\boldsymbol{\varepsilon}^e \;, \tag{6}$$

where $\mathbf{C}$ is interpreted as the stiffness tensor (positively defined, double symmetric, fourth order tensor). Stored elastic energy function W_S is the following function of orthotropic invariants of elastic strain tensor $\boldsymbol{\varepsilon}^e$:

$$\begin{aligned} W_S = {} & \frac{1}{2}\sum_{i=1}^{3} \bar{a}_i \left(\mathrm{tr}\boldsymbol{\varepsilon}^e \tilde{\mathbf{M}}_i\right)^2 + \bar{b}_1 \mathrm{tr}\boldsymbol{\varepsilon}^e \tilde{\mathbf{M}}_2 \mathrm{tr}\boldsymbol{\varepsilon}^e \tilde{\mathbf{M}}_3 + \bar{b}_2 \mathrm{tr}\boldsymbol{\varepsilon}^e \tilde{\mathbf{M}}_3 \mathrm{tr}\boldsymbol{\varepsilon}^e \tilde{\mathbf{M}}_1 + \bar{b}_3 \mathrm{tr}\boldsymbol{\varepsilon}^e \tilde{\mathbf{M}}_1 \mathrm{tr}\boldsymbol{\varepsilon}^e \tilde{\mathbf{M}}_2 \\ & + \sum_{i=1}^{3} \bar{c}_i \mathrm{tr}\left(\boldsymbol{\varepsilon}^e\right)^2 \tilde{\mathbf{M}}_i , \end{aligned} \tag{7}$$

where $\bar{a}_i$, $\bar{b}_i$ and $\bar{c}_i$ are elasticity parameters (which are independent from coordinate system, contrary to the stiffness tensor components). An interpretation of material parameters from (7) is given in [6], together with relationships expressing $\bar{a}_i$, $\bar{b}_i$ and $\bar{c}_i$ via technical elasticity parameters (e.g. Young moduli and Poisson ratios). It is worth emphasizing that $\mathbf{C}$ is an orthotropic tensor function of second order parametric tensors $\tilde{\mathbf{M}}_i$. In general we assume that parametric tensors in (3) are different than these occurring in (7).

Due to the convenience of implementing the model in ABAQUS, yield condition is rewritten in the following form:

$$f_\alpha\left(\boldsymbol{\sigma}, \kappa_\alpha\right) = \frac{1}{2}\boldsymbol{\sigma}.\mathbf{P}_\alpha.\boldsymbol{\sigma} + \mathbf{p}_\alpha.\boldsymbol{\sigma} - \left(1 + L_\alpha \kappa_\alpha\right) = 0 \,, \tag{8}$$

where $\mathbf{P}_\alpha$ and $\mathbf{p}_\alpha$ are respectively tensor functions of the fourth and second order, depending on the parametric tensors $\mathbf{M}_i$ and material parameters defining plastic properties.

In standard theory of elastic-plastic materials formulated in the stress space, only such states of stress are allowed, for which $f_\alpha \leq 0$. If $f_\alpha < 0$, than state of stress is called elastic. Plastic part of the strain tensor is defined as associated with the yield functions (8) (or in the form (1)):

$$\dot{\boldsymbol{\varepsilon}}_\alpha^P = \dot{\gamma}_\alpha \left. \frac{\partial f_\alpha(\boldsymbol{\sigma}, \alpha)}{\partial \boldsymbol{\sigma}} \right|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^T} = \dot{\gamma}_\alpha \left(\mathbf{P}_\alpha . \boldsymbol{\sigma} + \mathbf{p}_\alpha\right) \equiv \dot{\gamma}_\alpha \mathbf{g}_\alpha \ , \tag{9}$$

where $\dot{\gamma}_\alpha > 0$ is so called plastic multiplayer. From (5) i (6), after differentiation with respect to time and substituting (8) we obtain:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}.\left(\dot{\boldsymbol{\varepsilon}} - \dot{\gamma}_\alpha \mathbf{g}_\alpha\right) \equiv \mathbf{C}_{ep} . \dot{\boldsymbol{\varepsilon}} \ , \tag{10}$$

where $\mathbf{C}_{ep}$ operator (double symmetrical fourth order tensor) we evaluate after obtaining result for $\dot{\gamma}_\alpha$. From the condition of conformity

$$\dot{\gamma}_\alpha \dot{f}_\alpha(\boldsymbol{\sigma}, \kappa_\alpha) = 0 \ , \quad \dot{\gamma}_\alpha > 0 \ , \tag{11}$$

with assumption that

$$\dot{\kappa}_\alpha = \dot{\gamma}_\alpha \mathbf{g}_\alpha . \mathbf{g}_\alpha \equiv \dot{\gamma}_\alpha \left\|\mathbf{g}_\alpha\right\|^2 , \tag{12}$$

we obtain plastic multiplayer as

$$\dot{\gamma}_\alpha = \frac{\mathbf{g}_\alpha . \mathbf{C} . \dot{\boldsymbol{\varepsilon}}}{\mathbf{g}_\alpha . \mathbf{C} . \mathbf{g}_\alpha + \kappa_\alpha \left\|\mathbf{g}_\alpha\right\|^2} \ . \tag{13}$$

Denominator in (13) is always positive, which meet the requirement $\dot{\gamma} > 0$ only if the numerator (13) is positive. After substituting (13) to (10) we obtain:

$$\mathbf{C}_\alpha^{ep} = \mathbf{C} - \frac{\left(\mathbf{g}_\alpha . \mathbf{C}\right) \otimes \left(\mathbf{C} . \mathbf{g}_\alpha\right)}{\mathbf{g}_\alpha . \mathbf{C} . \mathbf{g}_\alpha + L_\alpha \left\|\mathbf{g}_\alpha\right\|^2} \ . \tag{14}$$

The formal generalization of the proposed model is the adoption in (7) so called piecewise linear function. Then for every n-th interval of plastic variable $\dot{\kappa}_\alpha$ we assume different L_α.

For practical considerations we assume that the stress space is limited by two surfaces of the type (1). In such case always it is needed to solve the problem of evaluating gradient at the hyper-lines connecting these surfaces (and so-called corner points). In such case we assume the following linear combination of the two gradients:

$$\mathbf{g}(\boldsymbol{\sigma}) = \lambda \mathbf{g}_1(\boldsymbol{\sigma}) + (1-\lambda)\mathbf{g}_2(\boldsymbol{\sigma}), \quad \lambda \in (0,1), \tag{15}$$

where the scalar parameter λ selection depends on the adopted algorithm. If we assume *a priori* $\lambda = 1/2$, we get so called Koiter rule.

4. NUMERICAL IMPLEMENTATION

Constitutive relationship (10) is in the form of a "highly non-linear" differential equation which can be solved using modified Euler method (so called “forward” or “backward”). Therefore, (10) is replaced by the incremental equation of the form:

$$\Delta\boldsymbol{\sigma} = \mathbf{C}.\left(\Delta\boldsymbol{\varepsilon} - \Delta\gamma_\alpha \tilde{\mathbf{r}}\right) \equiv \tilde{\mathbf{C}}_{ep}.\Delta\boldsymbol{\varepsilon} \;, \tag{16}$$

where $\tilde{\mathbf{C}}_{ep}$ is a consistent with an algorithm integrating the constitutive relations. We assume that for all $t_n \in [0,T]$ we know $\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n$ (controlled deformation) and we want to calculate the stress state $\boldsymbol{\sigma}_{n+1}$ for t_{n+1}. It is assumed than that

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{trl} - \Delta\gamma_\alpha \mathbf{C}.\mathbf{g}_\alpha^{trl}, \tag{17}$$

where $\boldsymbol{\sigma}_{n+1}^{trl} = \mathbf{C}.\boldsymbol{\varepsilon}_{n+1}$ is so called trial elastic stress state and $\mathbf{g}_\alpha^{trl}$ is a gradient for $f_\alpha\left(\boldsymbol{\sigma}_{n+1}^{trl}, \kappa_\alpha\right)$.

It should be emphasized that the calculation of the multiplier and the tensor function $\mathbf{g}_\alpha^{trl}$, is significantly dependent on the numerical implementation. General method of implementation of such a model is presented in [7] on the example of the theory of elasto-plasticity with the Huber-Mises (isotropy) and Hill (orthotropy) yield conditions, while a detailed description of the numerical FEM implementation is given in the monograph [4]. In the case of the constitutive relations considered in this work we can proceed similarly as in the above articles in order to obtain the operator $\tilde{\mathbf{C}}_{ep}$. The most important step is to calculate the plastic multipliers $\Delta\gamma_\alpha$ from quadratic equation of the variable $\Delta\gamma_\alpha$, therefore, significantly different than for Hill's yield condition.

Implementation of the proposed model of the material in the program ABAQUS is the programming task in the FORTRAN language. In the subroutine UMAT it is needed to evaluate the stress and stiffness tensor at the end of the time step, what always is connected with constitutive relationship integration. This procedure requires the definition of stress and strain in the vector form according to the so-called Voigt notation, which is not convenient because of the need to write the representation of tensors in the bases and co-bases. It should be pointed out that in program ABAQUS, the standard algorithms are available, e.g.: the Newton-Raphson and RIKS for solving a nonlinear equations of construction equilibrium.

5. NUMERICAL TEST ON STRUCTURAL LEVEL FOR TWO HOFFMAN CRITERIA – SHEARING OF THE MASONRY WALL

Shearing of the masonry wall is modelled as the three-dimensional test. The main goal of this test is to predict the failure mode. The example presented here is a masonry wall 1.05[m] long and 0.8[m] high with a rectangular opening (EF=HG=0.2[m], EH=FG=0.3[m]) placed 0.4[m] from axis 2 and 0.25[m] from axis 1, cf. fig.1. The thickness of the wall is assumed equal to 0.12[m]. Shearing of the wall is realized by displacement boundary conditions: for edge DCD'C' the displacements in direction 1 and 2 of all nodes are assumed equal to zero, while for edge ABA'B' the displacement u_1=-0.1[mm]. For the remaining edge surfaces the zero stress boundary conditions are assumed.

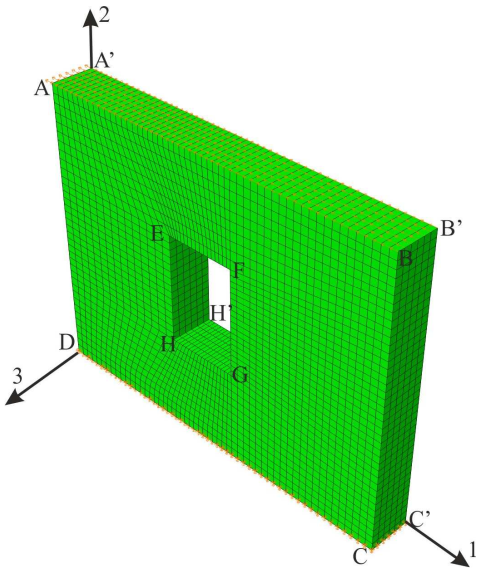

Fig. 1. Schematic view of the masonry wall exposed to the shearing. FEM mesh, boundary conditions and geometry

The material constitutive model adopted in this case is the one described in section 3 (simplified model with Hoffman criterion). The material parameters used for simulation are presented in tab.1. It was assumed also that L_α =0, so we are dealing with elastic-perfectly plastic material.

Tab. 1. Material properties

Elastic properties			Inelastic properties f_1			Inelastic properties f_2		
E_1 8 GPa	E_2 7.2GPa	E_3 6.4GPa	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]	[MPa]
G_{12} 3 GPa	G_{13} 2.5GPa	G_{23} 2 GPa	Y_{c1} 1.05	Y_{c2} 1.05	Y_{c3} 16.4	Y_{c1} 1.0	Y_{c2} 1.0	Y_{c3} 1.6
ν_{12} 0.15	ν_{13} 0.25	ν_{23} 0.2	Y_{t1} 0.25	Y_{t2} 0.25	Y_{t3} 0.25	Y_{t1} 1.0	Y_{t2} 1.0	Y_{t3} 1.0
$\nu_{21} = \nu_{12}E_2/E_1$	$\nu_{31} = \nu_{13}E_3/E_1$	$\nu_{32} = \nu_{23}E_3/E_2$	k_{12} 0.4	k_{13} 0.4	k_{23} 0.4	k_{12} 0.4	k_{13} 0.4	k_{23} 0.4

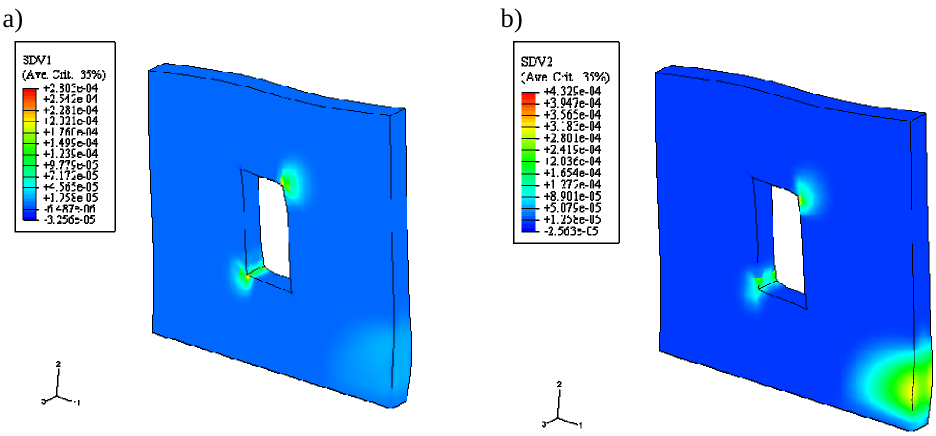

Fig. 2. Contour plots of the plastic strain components: a) ε_{11}^{p} *, b)* ε_{22}^{p} *.*

Displaying the plastic strain gives a good indication of the “cracked” areas in the model. Figure 2 gives a contour plot of the plastic strain components with strain localization near corners F and H of the opening. Such behaviour is observed in experiments and in other numerical simulations, cf.[6].

6. FINAL REMARKS

The continuum structural model for the analysis of masonry structures is proposed in the paper. Constitutive relations are established in the framework of the mathematical elastoplasticity theory of small displacements. Based on the new orthotropic failure criterion that was earlier proposed by the authors, the model includes a generalization of the well known Hoffman failure criterion. That criterion is chosen in order to present the implementation of the model into the finite element program ABAQUS. The test of the proposed incremental-iterative algorithm of the finite element method for an anisotropic continuum is presented in the paper on the example of the masonry wall shearing. At present, the model may be useful in the prediction of the load capacity of masonry structures since the implementation of the softening into finite element program is currently in progress.

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