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Pochodno-całka Caputo niecałkowitego rzędu, metoda oczkowa, stan nieustalony, zasada wzajemności, zastępcze źródło prądowe, zastępcze źródło napięciowe

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ANALYSIS OF FRACTIONAL ELECTRICAL CIRCUITS IN TRANSIENT STATES

Using the Caputo definition of fractional derivative-integral it is shown that the mesh method can be also applied to analysis of fractional linear electrical circuits in transient state. Using the mesh method it is shown that the reciprocity theorem is also valid for fractional linear circuits in transient state. The classical Thevenin theorem and Norton theorem are extended for fractional linear electrical circuits.

ANALIZA LINIOWYCH OBWODÓW ELEKTRYCZNYCH NIECAŁKOWITEGO RZĘDU W STANIIE NIEUSTALONYM

Korzystając z pochodno-całki Caputo rzędu niecałkowitego uogólniono metodę oczkową na rozgałęzione obwody elektryczne rzędu niecałkowitego w stanach nieustalonych. Wykazano, że: 1) zasada wzajemności jest również prawdziwa dla liniowych obwodów elektrycznych niecałkowitego rzędu, 2) twierdzenie o zastępczym źródle napięciowym i twierdzenie o zastępczym źródle prądowym są również prawdziwe dla liniowych obwodów elektrycznych niecałkowitego rzędu w stanie nieustalonym

1. ITRODUCYION

The mesh method and the node method are the basic methods of analysis of linear electrical circuits [1, 4, 6]. In this analysis the reciprocity theorem, the classical Thevenin theorem and Norton theorem play an important role.

Recently a dynamical development of the fractional linear systems theory can be observed [11-13]. An overview of state of art in positive systems theory is given in the monograph [10]. The stability of fractional linear discrete-time and continuous-time systems has been investigated in [2, 3, 5, 8, 9].

In this paper it will be shown that the mesh method can be applied to analysis of fractional linear circuits in transient state and the classical reciprocity theorem, Thevenin theorem and Norton theorem will be extended for fractional linear circuits in transient state.

The paper is organized as follows. In section 2 the basic Caputo definition of the fractional derivative-integral and the fractional state equation and its solution are recalled. An

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extension of the mesh method for the fractional linear circuits in transient state are given in section 3.In section 4 the reciprocity theorem is extended for fractional linear circuits in transient state and in section 5 the classical Thevenin theorem and the Norton theorem are extended for fractional linear circuits in transient state. Concluding remarks are given in section 6.

2. DERIVATIVE-INTEGRAL OF FRACTIONAL ORDER AND SOLUTION TO STATE EQUATIONS OF LINEAR SYSTEMS

In this paper the following Caputo definition of the derivative-integral of fractional order will be used [10-13]

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \,,\; n-1 < \alpha < n \,,\; n \in N = \{1,2,...\} \tag{1}$$

where

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \,,\; \mathrm{Re}(x) > 0 \tag{2}$$

is the gamma function and

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n} \tag{3}$$

is the classical *n* order derivative.
It is easy to show [10-12] that the Laplace transform of (1) has the form

$$\mathcal{L}\left[\frac{d^{\alpha} f(t)}{dt^{\alpha}}\right] = \int_0^{\infty} \frac{f^{\alpha}(t)}{dt^{\alpha}} e^{-st} d\tau = s^{\alpha} F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0+) \tag{4}$$

where $F(s) = \mathcal{L}[f(t)]$.

Consider the linear continuous-time of the fractional order α, $0 < \alpha < 1$ system described by the state equation

$$\frac{d^{\alpha} x(t)}{dt^{\alpha}} = Ax(t) + Bu(t) \tag{5}$$

where $x(t) \in R^n$, $u(t) \in R^m$ are the state and input vectors and $A \in R^{n \times n}$, $B \in R^{n \times m}$ (the set of real $n \times m$ matrices).

Theorem 1. Solution of the equation (5) satisfying the initial condition $x(0) = x_0$ has the form

$$x(t) = \Phi_0(t) x_0 + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau \tag{6}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \quad 0 < \alpha < 1 \tag{7}$$

3. ANALYSIS OF THE FRACTIONAL LINEAR CIRCUITS BY THE USE OF THE MESH METHOD

Let the current $i_C(t)$ in the condensator with the capacity C be the α order derivative of its charge $q(t)$

$$i_C(t) = \frac{d^\alpha q(t)}{dt^\alpha} \tag{8a}$$

Using $q(t) = Cu_C(t)$ we obtain

$$i_C(t) = C\frac{d^\alpha u_C(t)}{dt^\alpha} \tag{8b}$$

where $u_C(t)$ is the voltage on the condensator.
Similarly, let the voltage $u_L(t)$ on coil (inductor) with the inductance L be the β order derivative of its magnetic flux $\Psi(t)$

$$u_L(t) = \frac{d^\beta \Psi(t)}{dt^\beta} \tag{9a}$$

Taking into account that $\Psi(t) = Li_L(t)$ we obtain

$$u_L(t) = L\frac{d^\beta i_L(t)}{dt^\beta} \tag{9b}$$

where $i_L(t)$ is the current in the coil (inductor).
Using the relation (4) for (8b) (n = 1) we obtain

$$I_C(s) = s^\alpha CU_C(s) - Cs^{\alpha-1}i_C(0+), \quad 0 < \alpha < 1 \tag{10}$$

where $I_C(s) = \mathcal{L}[i_C(t)]$ and $U_C(s) = \mathcal{L}[u_C(t)]$.
Similarly, using the relation (4) for (9b) we obtain

$$U_L(s) = s^\beta LI_L(s) - Ls^{\beta-1}i_L(0+), \quad 0 < \beta < 1 \tag{11}$$

where $U_L(s) = \mathcal{L}[u_L(t)]$ and $I_L(s) = \mathcal{L}[i_L(t)]$.
Impedance of the series connection of the resistance R, the capacity C and inductance L described by the relations (8b) and (9b)) will be called the operator impedance.
To simplify the considerations we shall assume that:

1) initial conditions are zero: $i_C(0+) = 0, \quad u_L(0+) = 0$,
2) all Laplace transforms $U_C(s)$ and $I_C(s)$ of the condensators are related by

$$U_C(s) = \frac{1}{s^\alpha C}I_C(s) \tag{12a}$$

3) all Laplace transforms $U_L(s)$ and $I_L(s)$ of the coils are related by

$$U_L(s) = s^\beta LI_L(s) \tag{12b}$$

First we shall show the essence of the mesh method of the following electrical circuit (Fig. 1) with given resistances $R_1, R_2, R_3, R_4, R_5, R_6$, capacitances C_2, C_3, C_6, inductances L_1, L_5, L_6 and source voltages e_1, e_2, e_4, e_6. Let $I'_1(s), I'_2(s), I'_3(s)$ be the Laplace transforms of the mesh currents $i'_1(t), i'_2(t), i'_3(t)$

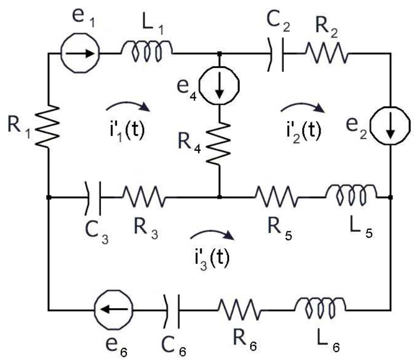

Fig. 1. Electrical circuit.

Using the Kirchhoff's voltage law and the relations (12) for the circuit we obtain the equations

$$
\begin{aligned}
E_1(s)+E_4(s) &= (R_1+s^{\beta}L_1)I'_1(s)+R_4[I'_1(s)-I'_2(s)]+\left(R_3+\frac{1}{s^{\alpha}C_3}\right)[I'_1(s)-I'_3(s)]\\
&= Z_{11}(s)I'_1(s)-Z_{12}(s)I'_2(s)-Z_{13}(s)I'_3(s)\\
E_2(s)-E_4(s) &= (R_2+\frac{1}{s^{\alpha}C_2})I'_2(s)+(R_5+s^{\beta}L_5)[I'_2(s)-I'_3(s)]+R_4[I'_2(s)-I'_1(s)]\\
&= -Z_{21}(s)I'_1(s)+Z_{22}(s)I'_2(s)-Z_{23}(s)I'_3(s)\\
E_6(s) &= (R_6+s^{\beta}L_6+\frac{1}{s^{\alpha}C_6})I'_3(s)+\left(R_3+\frac{1}{s^{\alpha}C_3}\right)[I'_3(s)-I'_1(s)]+(R_5+s^{\beta}L_5)[I'_3(s)-I'_2(s)]\\
&= -Z_{31}(s)I'_1(s)-Z_{32}(s)I'_2(s)+Z_{33}(s)I'_3(s)
\end{aligned}
\tag{13}
$$

where $E_k(s)=\mathcal{L}[e_k(t)]$, k = 1,2,4,6; $Z_{11}(s)=R_1+R_3+R_4+s^{\beta}L_1+\frac{1}{s^{\alpha}C_3}$,

$Z_{12}(s)=Z_{21}(s)=R_4$, $Z_{13}(s)=Z_{31}(s)=R_3+\frac{1}{s^{\alpha}C_3}$, $Z_{22}(s)=R_2+R_4+R_5+s^{\beta}L_5+\frac{1}{s^{\alpha}C_2}$,

$Z_{23}(s)=Z_{32}(s)=R_5+s^{\beta}L_5$, $Z_{33}(s)=R_3+R_5+R_6+s^{\beta}(L_5+L_6)+\frac{1}{s^{\alpha}C_3}+\frac{1}{s^{\alpha}C_6}$.

Equations (13) can be written in the form

$$E'(s)=Z(s)I'(s) \tag{14a}$$

where

$$E'(s)=\begin{bmatrix}E_1(s)+E_4(s)\\E_2(s)-E_4(s)\\E_6(s)\end{bmatrix},\quad Z(s)=\begin{bmatrix}Z_{11}(s) & -Z_{12}(s) & -Z_{13}(s)\\ -Z_{21}(s) & Z_{22}(s) & -Z_{23}(s)\\ -Z_{31}(s) & -Z_{32}(s) & Z_{33}(s)\end{bmatrix},\quad I'(s)=\begin{bmatrix}I'_1(s)\\I'_2(s)\\I'_3(s)\end{bmatrix} \tag{14b}$$

Taking into account that $\det Z(s)\neq 0$ we may find from the equation (14a) the vector $I'(s)$

$$I'(s)=Z^{-1}(s)E'(s) \tag{15}$$

Applying the inverse Laplace transform ($\mathcal{L}^{-1}$) to $I'(s)$ we may find the mesh currents $i'_1(t), i'_2(t), i'_3(t)$, and next from the relations

$$\begin{aligned} &i_1(t) = i'_1(t), \quad i_2(t) = i'_2(t), \quad i_3(t) = i'_3(t) - i'_1(t), \\ &i_4(t) = i'_1(t) - i'_2(t), \quad i_5(t) = i'_2(t) - i'_3(t), \quad i_6(t) = i'_3(t) \end{aligned} \tag{16}$$

branch currents $i_k(t)$, k = 1,…,6 in transient state.

Remark. Choosing as the state variables (the components of the state vector $x(t)$) voltages across the condensators and currents in the coils we may write for the circuit (Fig. 1) the state equation (5), where the source voltages are the components of $u(t)$ and entries of the matrices A, B depend on the resistances, capacitances and inductances of the circuit. Using the solution (6) of the equation (5) we may find the transient values of the voltages across the condensators and of the currents in coils of the circuit.

In general case of n-mesh linear electrical circuit we obtain the equation (14a), where

$$E'(s) = \begin{bmatrix} E'_1(s) \\ E'_2(s) \\ \vdots \\ E'_n(s) \end{bmatrix}, \quad Z(s) = \begin{bmatrix} Z_{11}(s) & -Z_{12}(s) & \dots & -Z_{1n}(s) \\ -Z_{21}(s) & Z_{22}(s) & \dots & -Z_{2n}(s) \\ \vdots & \vdots & \dots & \vdots \\ -Z_{n1}(s) & -Z_{n2}(s) & \dots & Z_{nn}(s) \end{bmatrix}, \quad I'(s) = \begin{bmatrix} I'_1(s) \\ I'_2(s) \\ \vdots \\ I_n(s) \end{bmatrix} \tag{17}$$

$E'_k(s)$, k = 1,2,…,n is the algebraic sum of the Laplace transforms of sources voltages in the k-th mesh (with + we take the transform if its direction is consistent with the direction of the mesh current and with – if the direction is opposite). $Z_{kk}(s)$, k = 1,2,…,n is the sum of the operator impedances of all branches belonging to the k-th mesh and $Z_{kl}(s)$, k, l = 1,2,…,n is the operator impedance of the branch belonging to the k-th mesh and l-th mesh. $I'_k(s)$, k = 1,2,…,n is the Laplace transform of the k-th mesh current.

Knowing $E'(s)$ and $Z(s)$ from the equation (15) we may find $I'(s)$ and using the inverse Laplace transform we may find the mesh currents $i'_k(t)$, k = 1,2,…,n and next the branch currents of the circuit.

4. RECIPROCITY THEOREM

Consider a fractional linear circuit composed of resistances, capacitances ,inductances and one source voltage $e(t)$ in transient state. We choose the linearly independent meshes in such way that the source voltage belongs to the k-th mesh and let $i(t)$ be the current in a branch belonging only to the l-th mesh (Fig. 2a).

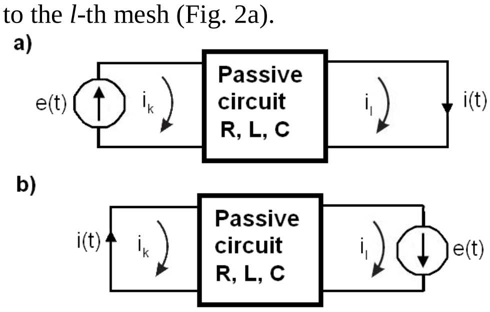

Fig. 2. Electrical circuits.

Applying to the circuit the mesh method from (15) we obtain

$$I_l(s) = \frac{C_{lk}(s)}{\det Z(s)} E(s) \text{ for } k, l = 1,2,\dots,n \tag{18}$$

where $I_l(s) = \mathcal{L}[i_l(t)]$, $E(s) = \mathcal{L}[e(t)]$, $C_{lk}(s) = (-1)^{k+l} M_{lk}(s)$, $M_{lk}(s)$ is the minor obtained from the matrix $Z(s)$ by deleting its l-th row and its k-th column.

Now we interchange the places of the source voltage $e(t)$ and the observation point of the current $i(t)$ (Fig. 2b). Applying to the circuit from Fig. 2b the mesh method from (15) we obtain

$$I_k(s) = \frac{C_{kl}(s)}{\det Z(s)} E(s) \tag{19}$$

From symmetry of the matrix $Z(s)$ it follows that $C_{kl}(s) = C_{lk}(s)$ and this implies $I_k(s) = I_l(s)$. Therefore, the following reciprocity theorem has been proved.

Theorem 2. The ratio of a single source voltage at one point to observed branch current at another one in any linear fractional electrical circuit in transient state is invariant with respect to an interchange of the points of excitation and observation.

Example 1. Consider a linear fractional electrical circuit (Fig. 3) with given resistances R_1, R_2, R_3, capacity C, inductance L and source voltage $e(t)$. The source voltage belongs to the first mesh and $i(t)$ is the current equal to the second mesh current $i'_2(t)$.

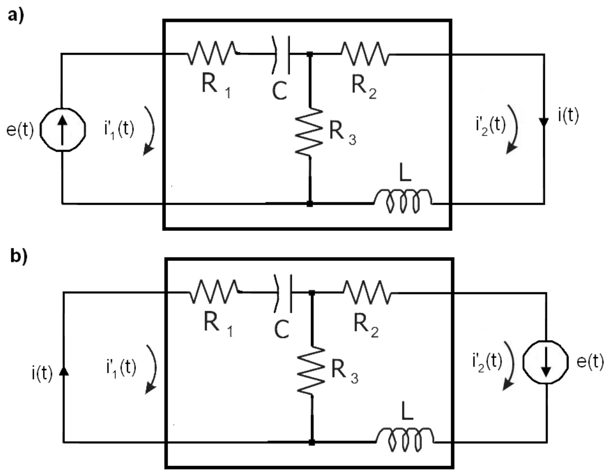

Fig. 3. Electrical circuit with single source voltage.

Equation (14) for the circuit has the form

$$\begin{bmatrix} E(s) \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + R_3 + \dfrac{1}{s^\alpha C} & -R_3 \\ -R_3 & R_2 + R_3 + s^\beta L \end{bmatrix} \begin{bmatrix} I'_1(s) \\ I'_2(s) \end{bmatrix} \tag{20}$$

From (20) we have

$$I'_2(s) = \frac{s^\alpha R_3 C}{\Delta(s)} E(s) \tag{21}$$

where

$$\Delta(s) = s^\alpha s^\beta LC(R_1 + R_3) + s^\alpha C[R_2(R_1 + R_3) + R_1 R_3] + s^\beta L + R_2 + R_3 \tag{22}$$

Similarly, equation (14) for the circuit from Fig. 3b has the form

$$\begin{bmatrix} 0 \\ E(s) \end{bmatrix} = \begin{bmatrix} R_1 + R_3 + \dfrac{1}{s^\alpha C} & -R_3 \\ -R_3 & R_2 + R_3 + s^\beta L \end{bmatrix} \begin{bmatrix} I'_1(s) \\ I'_2(s) \end{bmatrix} \tag{23}$$

From (20) we have

$$I'_1(s) = \frac{s^\alpha R_3 C}{\Delta(s)} E(s) \tag{24}$$

From comparison of (21) and (22) we obtain $I'_2(s) = I'_1(s)$.

5. EQUIVALENT VOLTAGE SOURCE THEOREM AND EQUIVALENT CURRENT SOURCE THEOREM

Consider fractional linear electrical circuit composed of resistances, capacitances, inductances and source voltages. The circuit can be divided in active part A and passive part P. Both parties are connected in the way shown on Fig. 4a.

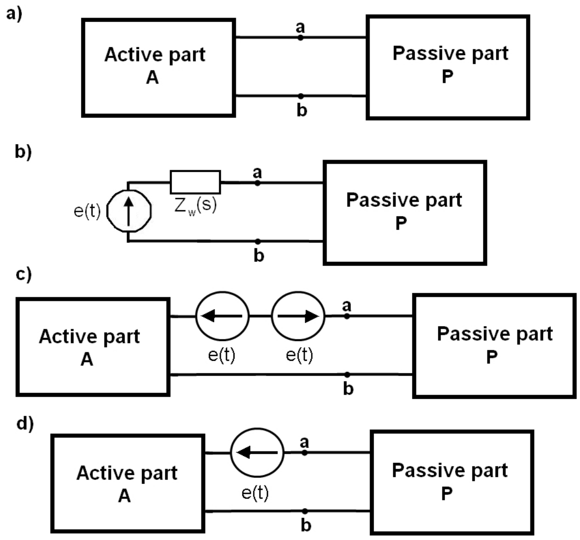

Fig. 4. Connection of active part A and passive part P.

After disconnecting the passive part P:

1) we register the voltage $u_0(t)$ between the points a-b in transient state,
2) we register the current $i_z(t)$ in transient state when the points a-b are short circuit,

3) we calculate the equivalent operator impedance $Z_w(s)$ of the active part A when all source voltages are zero.

We shall show that the active part A is equivalent to (ideal) voltage source $e(t) = u_0(t)$ connected in series with the operator impedance $Z_w(s)$ .To do this we switch on two voltage sources with opposite directions $e(t)$ (Fig. 4c). By assumption the fractional circuit is linear and we may use the superposition principle. Currents and voltages in transient state in the circuit shown on Fig. 4c are the sum of the suitable currents and voltages in the circuits shown on Fig. 4d and Fig. 4b.The voltage between the points a-b on Fig. 4d is equal zero and all currents and voltages in the passive part P are equal zero. The voltages and current in part P shown on Fig. 4a and Fig. 4b are the same. This completes the proof of the following theorem.

Theorem 3. Active part of any fractional linear system in transient state is equivalent to voltage source $e(t)$ connected in series with the operator impedance $Z_w(s)$ (Fig. 4b).

Example 2. The electrical circuit shown on Fig. 5 we divide into the active part A and the passive part P .

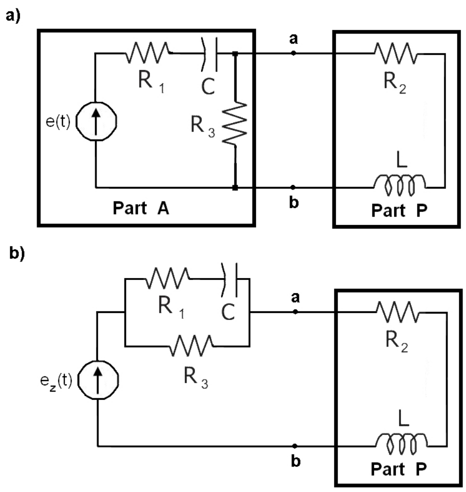

Fig. 5. Connection of the active part A and the passive part P.

The active part is equivalent to the voltage source $e_z(t)$ connected in series with the operator impedance $Z_w(s)$, where $e_z(t)$ is equal to the voltage on the resistance R_3 when the passive part P is disconnected and the operator impedance is

$$Z_w(s) = \frac{R_3(R_1 + \frac{1}{s^\alpha C})}{R_1 + R_3 + \frac{1}{s^\alpha C}} \tag{25}$$

Using the well-known equivalence of the voltage source and current source we obtain the following theorem.

Theorem 4. Active part of any fractional linear system in transient state is equivalent to current source $i_z(t)$ connected in parallel with the operator impedance $Z_w(s)$ (Fig. 6)

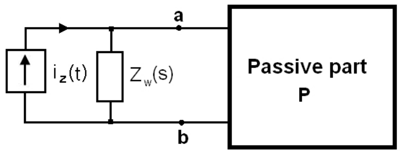

Fig. 6. Equivalent current source

where $i_z(t)$ is the current when the points a-b are short circuit and it is related to the source voltage $e_z(t)$ by the equality

$$\mathcal{L}[i_z(t)] = \frac{\mathcal{L}[e_z(t)]}{Z_w(s)} \tag{26}$$

Theorem 4 can be also proved in a similar way as the Theorem 3.

6. CONCLUDING REMARKS

Using the Caputo definition of the fractional derivative-integral and the Kirchhoff's laws it has been shown that the mesh method can be also applied to analysis of fractional linear circuits composed of resistances, capacitances, inductances and voltage (current) sources in transient state. In a similar way it can be shown that node method can be also applied to analysis of the fractional linear electrical circuits. Using the mesh method it has been shown that the reciprocity theorem is also valid for fractional linear circuits in transient state. The classical Thevenin theorem and Norton theorem have been extended for the fractional linear electrical circuits in transient state. These considerations can be extended for fractional linear electrical circuits with nonzero initial conditions.

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