Non-linear systems,
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OPTIMAL CONTROL FOR A PLANTS CONSISTING OF TWO MASSES AND CONNECTED WITH A NON-LINEAR SPRING

Problem of optimal system controlling a motion of the plant consisting of two-masses connected with a non-linear spring is considered in the paper. For example such plants can be motorcars with trailer or dumb barge pulling by ship. The irregular motion of elements of such plants influences negative on consume energy. In addition, because of safety, the speed of motion should be limited. As a criterion of quality we take the energy of the error signals. Till now the two mass problem (benchmark problem) was considered for linear cases only. The novelty of our work is generalisation of this problem (using the describing function method) for non-linear cases. The H_∞ control theory (robust control) is firstly adapted to cope with non-linear plants. High effectiveness of the optimal controller has been confirmed by computer simulation in MATLAB.

OPTYMALNE STEROWANIE OBIEKTAMI SKŁADAJĄCYMI SIĘ Z DWU MAS I POŁĄCZONYMI NIELINIOWĄ SPRĘŻYNĄ

W pracy rozważany jest układ sterujący, optymalizujący ruch obiektu składającego się z dwóch elementów o znacznych masach połączonych nieliniową sprężyną. Obiektami takimi są np. samochody z przyczepami lub barki wodne ciągnięte przez statki. Nieregularności ruchu elementów takich obiektów powodują duże straty energii w napędzie i, ze względu na bezpieczeństwo, zmuszają do ograniczania prędkości ruchu, co wydłuża czas trwania transportu. Kryterium jakości jest zużycie energii określone sygnałem błędu. Dla znalezienia optymalnego regulatora przeprowadzono harmoniczną linearyzację układu i dla zlinearyzowanego równania zastosowano metody optymalizacji bazujące na metodach przestrzeni Banacha H_∞ i H_2 (robust control). Wysoka skuteczność regulatora wyznaczonego opisaną metodą potwierdzona została symulacją komputerową. Dodatkowym efektem zastosowanego sterowania jest zapewnienie stabilności układu.

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1. INTRODUCTION

The problem of H_∞ optimal control for the plant consisting from two-masses m_1, m_2 and connected with a non-linear spring is considered in the paper. Till now the two mass problem (benchmark problem) was considered for linear cases only. The novelty of our work is a generalization of this problem (using the describing function method) for non-linear cases. The H_∞ control theory is firstly adapted to cope with non-linear plants. Up to this time the theorem is given assuming that the considered non-linear system is optimal in the sense of H_∞ norm criterion if corresponding harmonically linearized system is optimal in the same sense. This fact enables one, by utilization of describing function method, to bring the non-linear two-mass dynamical model to the linear approximation form and thereby to apply H_∞-type procedures.

Let us consider a non-linear dynamical model as in Fig. 1 given by the following set of equations:

$$m_1\, x_1''(t) + f(x_1(t) - x_2(t)) = u(t) \tag{1}$$

$$m_2\, x_2''(t) - f(x_1(t) - x_2(t)) = w(t), \tag{2}$$

where $x_1(t)$ and $x_2(t)$ are the positions of masses m_1 and m_2; the spring joining two masses is described by a non-linear operation f (for example $f(x) = |x|x^3$)

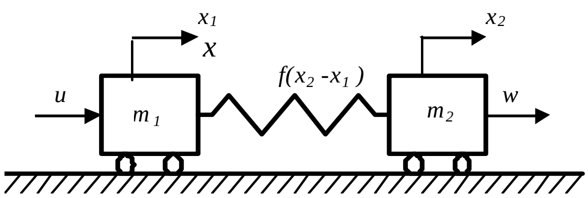

Figure 1. The exemplary plant in the non-linear benchmark problem

The signal $u(t)$ is a control force applied to mass m_1 and the plant is disturbed by $w(t)$.
We put $w(t) = 0$ and introduce the notations :

$$v_1 = x_2, \quad v_2 = x_1 - x_2, \quad v = [\mathrm{v}_1, v_2]^T,$$

The equations (1) ÷ (2) take the form:

$$v_1''(t) = \frac{1}{m_1} f(v_2(t)) \tag{3}$$

$$v_2''(t) = -\frac{m_1 + m_2}{m_1 m_2} f(v_2(t)) + \frac{1}{m_1} u(t) \tag{4}$$

In the paper the problem of two-masses (non-linear benchmark problem) tracking is tackled via H_∞-optimal control theory methods. Since the two-mass dynamical model is non-linear the theory is firstly adapted (using the describing function method) to cope with non-linear plants. The method of analysis of the nonlinear state feedback H_∞ optimal control was considered by A.J. van der Schaft [9]. The form of non-linear function *f*, in the considered plant, enable however to use the describing function method in simply way.

2. ADAPTATION OF H_∞ CONTROL THEORY TO THE CONSIDERED NON-LINEAR PROBLEM

Let us consider a tracking control system

$$\begin{cases} u = C_1(r) - C_2(r) \\ v = P(u) \end{cases} \tag{5}$$

where P is a non-linear dynamical operation, representing the plant (equations (3), (4)), which maps a set of signals with bounded energy for $t \in [0,\infty)$ into itself (Banach space $L^2 \rightarrow L^2$) and fulfills condition: $v(\cdot) = 0$ for $u(\cdot) = 0$. The main problem is that the signal *v* is to track a reference signal *r*. The plant input *u* is generated by passing *r* and *v* through linear controllers C_1 and C_2 respectively (as in Fig. 2).

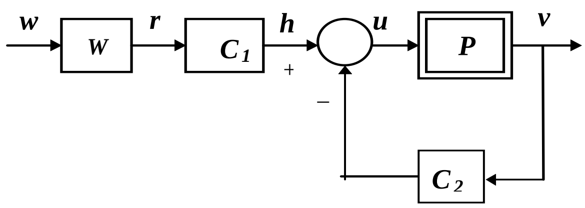

Fig. 2. Tracking system

It is postulated that *r* is not a known fixed signal but may be modeled as belonging to the class

$$R(W,m) = \{r : r = W(w) \text{ for some } w \in L_2, \|w\|_2 \leq m < \infty\} . \tag{6}$$

If the constant *m* is not determined exactly then the notation *R*(*W*) will be used.

The Laplace's transformations of signals *u*, *v*, *h*, *r* will be denoted by *u*(*s*), *v*(*s*), *h*(*s*), *r*(*s*) respectively. For the complex-value functions *x*(*s*) we use another, Hardy space H_2, besides L^2. Let H_∞ is a set of linear operations *F* for which there exist bounded in re(*s*) ≥ 0 transfer functions *F*(*s*). This set create a Banach space (H_∞ Hardy space). The subset of H_∞ consisting of rational functions with real coefficients will be denoted by RH_∞. We can formulate the following statement: if $F \in H_\infty$ and $x \in H_2$, then $F(x) \in H_2$, moreover

$$\|F\|_\infty = \sup\{ \|F(x)\|_2 : x \in H_2 , \|x\|_2 \leq 1 \}.$$

Since the tracking error signal is equal to $(r - v)$ the cost function is

$$\|z\|_2 = (\| r\text{-}v \|_2^2 + \| \rho u \|_2^2)^{1/2}, \tag{7}$$

where ρ is a non-negative weighting factor. Thus the tracking criterion takes here the form

$$I_2 = \min_{C_1, C_2} \left\{ \sup_{r \in R(W,m)} \left\{ \|z\|_2 \right\} \right\} \tag{8}$$

The minimization of the cost function (7) is equivalent to minimization of energy consumption in non-linear tracking control system (5).

For simplicity of our consideration let us assume at first that $C_1 = C_2$. This does not change the generality of the method because we can choose W (and the class of signals R(W)) that $W(C_1) = W_1(C_2)$. For example we can put $W = W_1(C_2(C_1^{-1}))$. We can transform formally the equations (5) to the form

$$\|r\text{-}v\|_2 = \|C_2^{-1}((C_2^{-1} + P)(r)\|_2 \tag{9}$$

$$\|u\|_2 = \|(C_2^{-1} + P)(r)\|_2 , \tag{10}$$

In connection with the operation P (mapping the space L^2 into itself) the describing function can be précised as follows [5]:

Let $v(t)$ be a response of a system described by the operation P to the signal $x(t) = N \sin(\omega t)$, a quotient of the symbolic value of the first harmonic of the output signal $v(t)$ to the amplitude of the input is called the describing function and is denoted by

$$\underline{P}(j\omega, N) = \frac{\frac{2}{T}\int_0^T v(t)\sin(\omega t)\,dt + j\frac{2}{T}\int_0^T v(t)\cos(\omega t)\,dt}{N} \tag{11}$$

Let us consider now a system which consists of two non-linear elements given by describing functions $\underline{P}_1$(jω,N) and $\underline{P}_2$(jω,N) and linear element with transfer function $K(s)$.

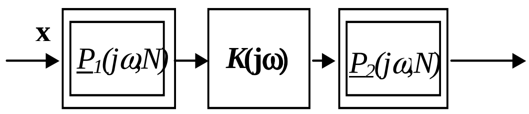

Fig. 3. Example of non-linear system

The resulting describing function $\underline{P}$(jω,N) can be expressed by formula:

$$\underline{P}(\mathrm{j}\omega,X) = \underline{P}_2(\mathrm{j}\omega,X|\underline{P}_1(\mathrm{j}\omega,X)||K(\mathrm{j}\omega)|)\ \underline{P}_1(\mathrm{j}\omega,X)\ K(\mathrm{j}\omega)\ .$$

Let $\underline{P}(s)$ be a describing function of some non-linear operation P. The approximate equations, which correspond to the equations (5), have the form

$$\begin{cases} u = C_1 r + C_2 v \\ v = \underline{P}u \end{cases} \tag{12}$$

System (12) can be treated as a system of linear differential equations.

The equivalent standard problem in H_∞ control theory (see [2], [3]), is defined by

$$y := \begin{bmatrix} r \\ v \end{bmatrix};\ K := \begin{bmatrix} C_1 & C_2 \end{bmatrix};\quad G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{13}$$

where

$$G_{11} := \begin{bmatrix} W \\ 0 \end{bmatrix};\qquad G_{12} := \begin{bmatrix} -P \\ \rho I \end{bmatrix}$$

$$G_{21} := \begin{bmatrix} W \\ 0 \end{bmatrix}\qquad G_{22} := \begin{bmatrix} 0 \\ P \end{bmatrix}$$

Since the methods of optimization in H_∞ space (13) refer to linear systems only the theorem which enables us to use these methods to optimization of a non-linear tracking systems is presented below. Let us assume that the linear operation C_2 is given by

$$\mathrm{C}_2(v(t)) = \int_0^t v(t-\tau)\mathrm{d}c(\tau)\,, \tag{14}$$

where $c(\tau)$ is a bounded variation function or an operation given by the transfer function $C_2(s)$. The equations (5) can be written in the form

$$\begin{cases} v(t) = P(u(t)) \\ u(t) = h(t) - \int_0^t v(t-\tau)\,\mathrm{d}c(\tau) \end{cases} \tag{15}$$

Theorem 1 (compare to [8]). Let an operation P mapping the Banach space L^2 into itself has a uniformly continuous and bounded derivative. If for a controller $C_2^* \in RH_\omega$ the expression

$$I_2' = \sup_{\omega\in R,N}\left(\left|\frac{1}{C(j\omega)}\right| + \rho\right)\cdot \left|\det\begin{bmatrix} re\dfrac{1}{C_2(j\omega)} + \dfrac{\partial re\underline{P}(j\omega)}{\partial imu(j\omega)} & -im\dfrac{1}{C_2(j\omega)} + \dfrac{\partial re\underline{P}(j\omega)}{\partial imu(j\omega)} \\ im\dfrac{1}{C_2(j\omega)} + \dfrac{\partial im\underline{P}(j\omega)}{\partial reu(j\omega)} & re\dfrac{1}{C_2(j\omega)} + \dfrac{\partial im\underline{P}(j\omega)}{\partial imu(j\omega)} \end{bmatrix}^{-1}\right| \qquad (16)$$

attains a finite minimum and if for the ω from this expression the number $\sup_{n\neq 1}\left|C_3\left(jn\varpi\right)\right|$ is sufficiently small then the control system described by equations (5) with the controller C_2^* is optimal in the sense of criterion (8). Moreover, under the same assumptions the controller C_2^* is optimal also for the approximate system (12) .

Conclusion. Let the assumptions of theorem 1 are fulfilled. If the controller $K^* = [C_1^*, C_2^*]$ is optimal in the sense of criterion (8) for the approximate system (12) then it is also optimal in the sense this criterion for the non-linear system (5).

Let the describing function $\underline{P}$ of some non-linear plant P be a rational function with real coefficients and analytic in the open right half-plane re $s > 0$ ($\underline{P} \in RH_\infty$). The system (12) can be transformed to the form of model matching (see Fig. 4) , where $T_i(s) \in RH_\infty$.

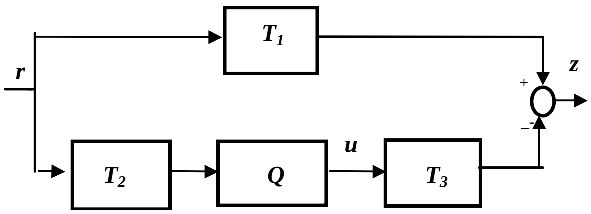

Fig. 4. Model-matching

In the case of stable describing function we can receive (compare (13)) that

$$K = -Q(I - G_{22}Q)^{-1} = -(I - G_{22}Q)^{-1}Q\,, \qquad T_1 = G_{11}\,,\ T_2 = G_{12}\,,\ \ T_3 = G_{21} \qquad (17)$$

In view of this fact the criterion equivalent to (8) is

$$I_2'' = \min_{Q\in RH_\infty}\left\|T_1 - T_2QT_3\right\|_\infty \ . \qquad (18)$$

For the two-mass control, the following problem can be solved too: compute an upper bound γ for I_2'' such that $\gamma - I_2''$ is less than a pre-specified tolerance; and then compute a $Q \in RH_\infty$ satisfying

$$\|T_1 - T_2\,QT_3\|_\infty \leq \gamma\,. \qquad (19)$$

Of course, such Q may not be optimal, but it can be calculated with any pre-assumed accuracy. Note that the considered methods can be generalized to include disturbances rejection problems [2], [3].

3. TWO-MASS MATHEMATICAL MODEL

Now we take under consideration the two-mass dynamical model given by the equations (3), (4).

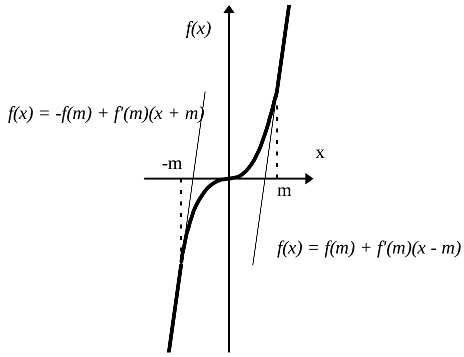

Fig. 5. Graph of function f(x)

We assume that the non-linear function (describing the action of spring between two mass) has the form

$$f(x)=\begin{cases}\sum_{k=1}^{n} ka_k m^{k-1}(x-m)+\sum_{k=1}^{n} a_k m^k & \text{for} \quad x \le m \\ a_1 x + a_2 x|x| + a_3 x^3 + \ldots + a_k x|x|^{k-1} & \text{for} \quad -m \le x \le m \\ \sum_{k=1}^{n} ka_k m^{k-1}(x-m)-\sum_{k=1}^{n} a_k m^k & \text{for} \quad x \le -m \end{cases} \tag{20}$$

Let us now find the describing function of the element given by equations (4) and (20). For this purpose we assume that output signal has the form

$$v_2(t) = v_2 \sin \omega t$$

so we get

$$u(t) = -m_1\omega^2 v_2 \sin \omega t + \frac{m_1+m_2}{m_2}\left(a_1 v_2 \sin \omega t + a_2 v_2^2 \sin \omega t + \ldots + a_k v_2^k \sin \omega t |\sin \omega t|^{k-1}\right)$$

Note that the following equation can be used

$$u(j\omega) = v_2(j\omega)\underline{P}^{-1}(j\omega, v_2).$$

Based on the above given definition of the describing function (11) we obtain

$$\left(\underline{P}^2(\mathrm{j}\omega,v_2)\right)^{-1}=2\frac{m_1m_2(\mathrm{j}\omega)^2+a_1(m_1+m_2)}{m_2} \qquad \text{for } k=1$$

$$\left(\underline{P}^2(\mathrm{j}\omega,v_2)\right)^{-1}=\frac{3\pi m_1m_2(\mathrm{j}\omega)^2+3\pi a_1(m_1+m_2)}{1.5\pi m_2}+\frac{8a_2v_2(m_1+m_2)}{1.5\pi m_2} \qquad \text{for } k=2$$

$$\left(\underline{P}^2(\mathrm{j}\omega,v_2)\right)^{-1}=\frac{12\pi m_1m_2(\mathrm{j}\omega)^2+12\pi a_1(m_1+m_2)}{6\pi m_2}+$$
$$+\frac{32a_2v_2(m_1+m_2)+9\pi a_3m_2v_2^2}{6\pi m_2} \qquad \text{for } k=3$$

$$\left(\underline{P}^2(\mathrm{j}\omega,v_2)\right)^{-1}=\frac{(60\pi a_1+160\pi a_2v_2)(m_1+m_2)}{30\pi m_2}+$$
$$+\frac{45\pi a_3m_2v_2^2+128a_4m_2v_2^2+60\pi m_1m_2(j\omega)^2}{30\pi m_2} \qquad \text{for } k=4$$

Then the first harmonic of input signal has the form

$$u(t)=u\sin\omega t=\left(\underline{P}^2(\mathrm{j}\omega,v_2)\right)^{-1}(v_2\sin\omega t)$$

i.e.

$$u=\left(\underline{P}^2(\mathrm{j}\omega,v_2)\right)^{-1}v_2\,.$$

Now we can write

$$\underline{P}^2(\mathrm{j}\omega,u)=\frac{m_2}{2(m_1m_2(\mathrm{j}\omega)^2+a_1(m_1+m_2))} \qquad \text{for } k=1$$

$$\underline{P}^2(\mathrm{j}\omega,u)=1{,}5\pi m_2(3\pi m_1m_2(\mathrm{j}\omega)^2+3\pi a_1(m_1+m_2)+8a_2v_2(m_1+m_2))^{-1} \qquad \text{for } k=2 \quad (21)$$

$$\underline{P}^2(\mathrm{j}\omega,u)=6\pi m_2(12\pi m_1m_2(\mathrm{j}\omega)^2+12\pi a_1(m_1+m_2)+32a_2v_2(m_1+m_2)+9\pi a_3m_2v_2^2)^{-1} \quad \text{for } k=3$$

$$\underline{P}^2(\mathrm{j}\omega,u)=\frac{A}{B+C(\mathrm{j}\omega)^2} \qquad \text{for } k=4$$

Where $A=30\pi m_2$;

$B=(60\pi a_1+160a_2v_2)(m_1+m_2)+45\pi a_3m_2v_2^2+128a_4m_2v_2^3$

$$C = 60\pi m_1 m_2$$

The function $v_2(u,\omega)$ in above formulas can be generated by the implicit functions respectively

$$u = v_2 \frac{3\pi(m_1 m_2 (\mathrm{j}\omega)^2 + a_1(m_1+m_2)) + 8a_2 v_2 (m_1+m_2)}{1{,}5\pi m_2} \qquad \textit{for } k = 2$$

$$u = \frac{v_2}{6\pi m_2}(12\pi m_1 m_2 (\mathrm{j}\omega)^2 + 12\pi a_1(m_1+m_2) + $$
$$+ 32\pi a_2 v_2 (m_2 + m_1) + 9\pi a_3 m_2 v_2^2) \qquad \text{for } k = 3 \qquad (22)$$

$$u = \frac{v_2}{30\pi m_2}(60\pi a_1(m_1+m_2) + 160 a_2 v_2 (m_1+m_2) + $$
$$+ 45\pi a_3 m_2 v_2^2 + 128 a_4 m_2 v_2^3 + 60\pi m_1 m_2 (j\omega)^2) \qquad \text{for } k = 4$$

It should be noticed that the describing functions $\underline{P}(\mathrm{j}\omega,v_0)$, for $k > 1$, are not real-rational. In the H_∞ control theory it is assumed that plant must be described by function from RH_∞. For this reason, the function $\underline{P}^2(\mathrm{j}\omega,u)$ will be approximated by real-rational function $\underline{P}^2_b(\mathrm{j}\omega,u)$. From many known methods of interpolation , the method of Lagrange's multipliers, used here, is presented bellow. Let points (v_{2i},ω_i) $i = 1, 2,\dots n$ fulfill the equations (22). We define a function

$$P_n^2(\omega) = \sum_{k=0}^{n} v_{2k} \frac{(\omega-\omega_0)(\omega-\omega_1)\dots(\omega-\omega_{k-1})(\omega-\omega_{k+1})\dots(\omega-\omega_n)}{(\omega_k-\omega_0)(\omega_k-\omega_1)\dots(\omega_k-\omega_{k-1})(\omega_k-\omega_{k+1})\dots(\omega_k-\omega_n)} \qquad (23)$$

So, the polynomial $\underline{P}^2_n(\omega)$ approximate the function $v_2(\omega,u)$. Substituting $\underline{P}^2_n(\omega)$ into equation (21) instead of $v_2(\omega,u)$ we get the describing function $\underline{P}^2_n(\mathrm{j}\omega,u)$ from R H_∞.

Example: Let us now find the describing functions $\underline{P}^2_n(\mathrm{j}\omega,u)$, $\underline{P}^1_n(\mathrm{j}\omega,u)$ from R H_∞ of the elements given by the equations (4) and (3) where the non-linear function has the form

$$f(x) = \begin{cases} -m^4 + 4m^3(x+m) & \text{for} \quad x \le -m \\ |x|x^3 & \text{for} \quad -m \le x \le m \\ m^4 + 4m^3(x-m) & \text{for} \quad x \ge m \end{cases}$$

From (21) for $k = 4$ we have

$$P^2(\mathrm{j}\omega, u) = \frac{15\pi m_2}{30\pi m_1 m_2 (\mathrm{j}\omega)^2 + 64 m_2 v_2^3} \tag{24}$$

where $v_2(\omega,u)$ is generated by the implicit function

$$64 m_2 v_2^4 - 30\pi m_1 m_2 \omega^2 v_2 - 15\pi m_2 u = 0 \tag{25}$$

Putting $m_1 = 1$, $m_2 = 2$, $u = 2$ into (25) we get

v_2 (0,2) =-1.1016,	v_2 (0.5,2) =-1.0232,
v_2 (1,2) =-0.7661,	v_2 (2,2) =-0.2493,
v_2 (4,2)=-0.0625,	v_2 (8,2) =-0.0156,
v_2 (16,2)=-0.0039,	v_2 (32,2)=-0.0010,
v_2 (100,2)= -0.0001 .	

Denoting by d(ω,u) the denominator of (24) we can write

d(0,2)=-171.1125,	d(0.5,2)= -184.2410,
d(1,2)= -246.0483,	d(2,2)= -755.9655,
d(4,2) -3.0160e+003,	d(8,2)=-1.2064e+004,
d(16,2)=-4.8255e+004,	d(32,2)=-1.9302e+005

The describing function $\underline{P}^2_n$ (jω,u) approximates the function (24) (by (23)) takes the form

$$\underline{P}_3^2 = -30\pi(0.2927(\mathrm{j}\omega)^3 + 197.0221(\mathrm{j}\omega)^2 + 72.1828(\mathrm{j}\omega) + 171.1136)^{-1} \tag{26}$$

The roots of the denominator of P^2_3(jω,u) are equal to -6.7275, -0.0018+0.0091j and -0.0018-0.0091j. For the equation (3) the describing function $\underline{P}^1_n$ (jω,u) takes the form

$$\underline{P}_3^1(\mathrm{j}\omega, 2) = -60\pi((\mathrm{j}\omega)^2 (0.2927(\mathrm{j}\omega)^3 + 197.022(\mathrm{j}\omega)^2 + 72.1828(\mathrm{j}\omega) + 171.1136)^{-1} \tag{27}$$

4. COMPUTATIONAL ALGORITHM

Now getting back to the tracking system (5), the equations (12) for the approximate plant take the form

$$u = C_1(r_1, r_2) - C_2(v_1, v_2) \tag{28}$$

$$v_1 = \underline{P}^1_n(\mathrm{j}\omega, u)u \tag{29}$$

$$v_2 = \underline{P}^2_n(\mathrm{j}\omega, u)u \tag{30}$$

where $\underline{P}^1_n$ and $\underline{P}^2_n$ are given by (26) and (27).

If we substitute

$$r = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} \; ; \; z = \begin{bmatrix} r_1 - v_1 \\ r_2 - v_2 \\ -\rho u \end{bmatrix}$$

$$Q = \begin{bmatrix} Q_1, & Q_2, & Q_3 \end{bmatrix}$$

$$T_1 = T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; ; \quad T_2 = \begin{bmatrix} -\underline{P}_n^1 \\ -\underline{P}_n^2 \\ -\rho I \end{bmatrix}$$

then the system (28), (29), (30) can be transformed to its equivalent model matching form (compare equations (17)). Using the algorithms for solving the model matching problem [2] we can find the optimal function Q^* and then the corresponding controller $K^*=[C_1^*, C_2^*]$. So, based on Theorem 1 the controller $K^*=[C_1^* \; C_2^*]$ is optimal also for the tracking system (5). The algorithm for the optimal controller Q^* in the sense of criterion (18), (19) has the following form:

Step 1. Compute Y and $\|Y\|_\infty$, where $Y:=(I-U_i\tilde{U}_i)T_1$ and $T_2=U_iU_0$ is inner-outer factorization, U_i is inner, $\tilde{U}_i = U_i(-s)$.

Step 2. Find an upper bound α_1 for α, where
$\alpha=\inf\{\gamma: \|Y\|_\infty< \gamma, \|Z\|_\infty<1, \text{dist}(R, R\,H_\infty)<1\}$,
Y_o=spectral factor of $\gamma^2- \tilde{Y}\,Y$,
$T_3Y_o^{-1}= V_{co}\,V_{ci}$ is coprime factorization, V_{co} co-outer, V_{ci} co-inner,
Z:= $\tilde{U}_i\,T_1Y_o^{-1}\,(I-\tilde{V}_{ci}\,V_{ci})$,
Z_{co}=co-spectral factor of $I-Z\tilde{Z}$, $R:=Z_{co}^{-1}\,\tilde{U}_i\,T_1\,Y_o^{-1}\,\tilde{V}_{ci}$.

Step 3. Select a trial value for γ in the interval $(\|Y\|_\infty, \alpha_1]$.

Step 4. Compute Z and $\|Z\|_\infty$.

Step 5. If $\|Z\|_\infty < 1$, continue; if not, increase γ and return to Step 4.

Step 6. Compute R and $\|\Gamma_R\|$. Then $\|\Gamma_R\| < 1$ if $\alpha < \gamma$, so increase or decrease the value of γ accordingly and return to Step 3. When a sufficiently accurate upper bound for α is obtained, continue.

Step 7. Find a matrix X in R H_∞ such that $\|R-X\|_\infty \le 1$.

Step 8. Solve $X = Z_{co}^{-1}\,U_o\,V_{co}$ for Q in R H_∞.

5. CONCLUSIONS

The presented theorem enables to apply the standard H_∞ methods to optimization of feedback control systems with non-linear plants. A linear, approximate two-mass dynamical model was obtained with the aid of describing function method (harmonic linearization). The performed MATLAB simulations confirmed the consistency between the given non-linear ship dynamics and its linear approximation. The presented algorithm make it possible to find a structure of the optimal controller for the tracking a preset two-mass trajectory. Note that the considered methods can be generalized to include disturbances rejection problems [2], [3].

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